From Self-Protection to Interaction Depletion: The Pressure-Hessian Sign in Curved and Interacting Vortex Tubes

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Abstract

In the preceding paper [2] we reduced the Navier-Stokes regularity question to the time-averaged sign of a single scalar observable $Q = e_2 \cdot H_{\mathrm{tf}} e_1$, and proved two vanishing theorems showing that $Q = 0$ for any z-translationally symmetric flow. The present paper computes Q for the first geometry that escapes these vanishing classes: a Burgers vortex tube with uniform curvature κ. The Frenet-frame metric $h_s = 1 + \kappa\rho\cos\varphi$ creates an off-diagonal strain perturbation $\Delta S_{xz} = -\gamma s\kappa/2$, uniform over the cross-section, which tilts the strain eigenframe and produces $Q \neq 0$ at $O(\kappa)$. However, this first nonzero mode is purely dipolar: $Q(\rho, \varphi) = 2.925\cos(\varphi - 31°)$ at $\rho = \sigma$, so the enstrophy-weighted cross-sectional average vanishes (Dipolar Oscillation Theorem). To obtain the first nonzero mean, we solve the $m = 1$ Poisson equation for the curvature-induced pressure perturbation $p_{c(\rho)} \cos\varphi$, compute the Hessian, and evaluate the $O(\kappa^2)$ cross-term between the eigenframe tilt and the pressure-Hessian response. The result is $\langle Q\rangle^{(2)} = +0.022 > 0$ at $\rho = \sigma$: the first net effect of curvature is *anti-depleting*. Curvature unlocks the observable but produces the wrong sign for regularity. Hence isolated tube curvature is not the depletion mechanism. We then compute Q for the simplest interacting configuration — a Burgers tube subject to the strain from a perpendicular vortex — and discover the *tidal gradient mechanism*: the spatial gradient of the external off-diagonal strain, when projected onto the cylindrical strain eigenbasis, creates an $m = 0$ component that survives azimuthal averaging. The enstrophy-weighted result obeys $\langle Q\rangle_\omega = C\gamma^2 \mathrm{Re}^2(\sigma/d)^3$ with $C \approx -0.55$, giving -5.52 at $\mathrm{Re} = 100$, $d = 10\sigma$: a robust depleting sign whose Re^2 scaling strengthens toward potential blow-up. The sign is controlled by a universal interaction kernel $F(\rho) < 0$ and correlates with stretching enhancement, providing the first constructive derivation of the depleting pressure-Hessian sign from vortex-vortex interaction. A self-consistency analysis shows that the blow-up scenario is *self-undermining*: self-consistent tube separation scales as $d/\sigma = \sqrt{\mathrm{Re}/2}$, the perturbation parameter decays as $\mathrm{Re}^{-3/2}$, and the depletion grows as $\sqrt{\mathrm{Re}}$ — all Z3-verified — so blow-up forces the system deeper into the depleting regime.

Keywords: Navier-Stokes equations, vortex tubes, curvature, pressure Hessian, alignment depletion, Burgers vortex, Frenet frame, tidal gradient, vortex interaction

1 Introduction

In a companion paper [1], we showed that scalar Sobolev methods cannot decide the Navier-Stokes regularity question: blow-up is consistent with every known inequality chain unless the stretching exponent sum $a + b$ is reduced from 4 to 2. In [2], we decomposed the stretching integral into geometric variables — strain eigenvalues λ_i and alignment weights $\alpha_i = (\xi \cdot e_i)^2$ — and showed that the missing exponent reduction localizes to a single scalar observable:

$$Q = e_2 \cdot H_{\mathrm{tf}} e_1$$

where H_{tf} is the trace-free pressure Hessian and e_1, e_2 are the most stretching and intermediate strain eigenvectors. The regularity question reduces, within the geometric framework, to whether $\langle Q \rangle < 0$ on average in high-enstrophy regions. Paper [2] then proved two vanishing theorems:

1. **Axisymmetric Vanishing:** $Q = 0$ in any axisymmetric flow with axial vorticity.
2. ***z*-Translation Vanishing:** $Q = 0$ for any flow $v(x, y, z) = v_\perp(x, y) + \gamma z \hat{z}$, regardless of in-plane structure.

The second theorem subsumes the first and eliminates *all* cross-sectional perturbations of straight tubes, including elliptically deformed Burgers vortices. The present paper asks the natural next question: what happens when we leave the vanishing class? The simplest symmetry-breaking geometry is a vortex tube with curvature. A Burgers vortex with uniform curvature κ has its axis along a circular arc rather than a straight line. The Frenet-frame metric introduces corrections at $O(\kappa)$ that break the z-translational symmetry, creating both off-diagonal strain and an $m = 1$ pressure perturbation. We first show that curvature alone is insufficient: a non-swirling vortex ring has $Q = 0$ *exactly* at all orders in κ (Ring Vanishing Theorem). The essential ingredient for $Q \neq 0$ is not curvature per se but the combination of curvature and axial velocity — the off-diagonal strain $\Delta S_{xz} = -v_s \kappa / 2$ vanishes when either factor is zero. For the bent Burgers tube, which has both ingredients ($\kappa > 0$ and $v_s = \gamma s$), we find three structural results: (i) $Q \neq 0$ at $O(\kappa)$ — the first model in our chain with genuinely nonzero Q; (ii) the first nonzero mode is purely dipolar ($m = 1$), so its cross-sectional average vanishes (Dipolar Oscillation Theorem); (iii) the first nonzero mean appears at $O(\kappa^2)$ from the product of the eigenframe tilt and the pressure-Hessian response, and its sign is *positive* — anti-depleting. These results have a clear physical interpretation: the pressure Hessian response to curvature *protects* the vortex tube's alignment with the stretching direction. Vortex rings have $Q = 0$ exactly — consistent with their observed stability — while bent tubes with axial stretching are actively anti-depleting. Finally, we compute Q for the simplest interacting configuration: a Burgers tube subject to the off-diagonal strain from a perpendicular vortex. A Fourier selection rule prevents direct coupling between curvature modes ($m = 1$) and tidal modes ($m = 2$). Instead, the *tidal gradient* of the external strain, when projected onto the cylindrical strain eigenbasis, creates an $m = 0$ component that survives azimuthal averaging. The enstrophy-weighted average obeys a scaling law $\langle Q \rangle_\omega = C \gamma^2 \mathrm{Re}^2 (\sigma / d)^3$ with $C \approx -0.55$, yielding a robust depleting sign that strengthens toward high Reynolds number and overwhelms the single-tube anti-depletion.

2 Strain Tensor in Curved Tube Coordinates

Consider a Burgers vortex whose axis has been bent with uniform curvature κ. We work in Frenet tube coordinates (ρ, φ, s) where ρ is the distance from the tube axis, φ is the azimuthal angle measured from the outward normal $\hat{n}$, and s is the arc length along the axis. The scale factors are:

$$h_\rho = 1, \quad h_\varphi = \rho, \quad h_s = 1 + \kappa\rho\cos\varphi \equiv H$$

The zeroth-order velocity field is the Burgers vortex:

$$v_\rho = -\gamma\rho/2, \quad v_\varphi = v_\theta(\rho), \quad v_s = \gamma s$$

where γ is the background strain rate and $v_\theta(\rho) = (\omega_0\sigma^2/2\rho)\left(1 - e^{-\rho^2/\sigma^2}\right)$ is the swirl velocity. Throughout this paper we work in units $\gamma = \sigma = 1$, so quantities with dimensions of inverse time appear as multiples of γ and lengths as multiples of σ. The single dimensionless control parameter is the vortex Reynolds number $\mathrm{Re} = \omega_0/\gamma$; we evaluate at $\mathrm{Re} = 100$. All numerical results carry implicit factors of γ and σ that are restored in the scaling laws below. The off-diagonal strain components coupling in-plane to axial directions are, in the orthonormal basis of the Frenet frame:

$$e_{\rho s} = \frac{H}{2}\frac{\partial}{\partial\rho}\left(\frac{v_s}{H}\right) + \frac{1}{2H}\frac{\partial v_\rho}{\partial s}$$

At zeroth order, $\partial v_\rho/\partial s = 0$ and $\partial(\gamma s/H)/\partial\rho = -\gamma s\kappa\cos\varphi/H^2$, giving:

$$e_{\rho s} = -\frac{\gamma s\kappa}{2}\cos\varphi + O(\kappa^2)$$

Similarly:

$$e_{\varphi s} = \frac{H}{2\rho}\frac{\partial}{\partial\varphi}\left(\frac{v_s}{H}\right) + \frac{\rho}{2H}\frac{\partial}{\partial s}\left(\frac{v_\varphi}{\rho}\right)$$

The second term vanishes (v_θ is s-independent). The first gives:

$$e_{\varphi s} = \frac{\gamma s\kappa}{2}\sin\varphi + O(\kappa^2)$$

2.1 Conversion to Cartesian coordinates

Converting to Cartesian coordinates at the reference point ($\hat{x} = \cos\varphi\,\hat{\rho} - \sin\varphi\,\hat{\varphi}$):

$$\Delta S_{xz} = e_{\rho s}\cos\varphi - e_{\varphi s}\sin\varphi = -\frac{\gamma s\kappa}{2}(\cos^2\varphi + \sin^2\varphi) = -\frac{\gamma s\kappa}{2}$$

$$\Delta S_{yz} = e_{\rho s}\sin\varphi + e_{\varphi s}\cos\varphi = 0$$

This is a remarkably clean result: the curvature creates a *uniform* off-diagonal strain $\Delta S_{xz} = -\gamma s\kappa/2$, independent of the azimuthal angle φ. The uniformity arises because the $\cos\varphi$ and $\sin\varphi$ contributions from $e_{\rho s}$ and $e_{\varphi s}$ combine to give a constant. We note that ΔS_{xz} vanishes at $s = 0$ (the symmetry point where the axial velocity $v_s = \gamma s$ is zero) and grows linearly with s. For a fluid element at position s on the tube, the strain perturbation is proportional to both the curvature κ and the axial velocity γs.

3 First-Order Q: Dipolar Oscillation

The zeroth-order strain eigenframe has:

$$\lambda_1 = -\gamma/2 + |S_{r\theta}|, \quad e_1 = (\hat{\rho} + \hat{\varphi})/\sqrt{2}$$

$$\lambda_2 = \gamma, \quad e_2 = \hat{z}$$

$$\lambda_3 = -\gamma/2 - |S_{r\theta}|, \quad e_3 = (\hat{\rho} - \hat{\varphi})/\sqrt{2}$$

where $S_{r\theta} = (v_\theta{}' - v_\theta/\rho)/2$ is the swirl shear. At high Reynolds number ($\omega_0 \gg \gamma$), we have $\lambda_1 \approx \omega_0/4$ and $\lambda_3 \approx -\omega_0/4$. The strain perturbation $\Delta S_{xz} = -\gamma s\kappa/2$ projects onto the eigenbasis as:

$$\Delta S_{12} = e_1 \cdot \Delta S \cdot e_2 = \Delta S_{xz} \cdot \frac{\cos\varphi - \sin\varphi}{\sqrt{2}}$$

$$\Delta S_{32} = e_3 \cdot \Delta S \cdot e_2 = \Delta S_{xz} \cdot \frac{\cos\varphi + \sin\varphi}{\sqrt{2}}$$

The first-order Q comes from the eigenframe perturbation (terms $B + C$ in standard perturbation theory):

$$Q^{(1)} = \frac{\Delta S_{12}}{\lambda_2 - \lambda_1}(H_+ - H_{zz}) + \frac{\Delta S_{32}}{\lambda_2 - \lambda_3} H_-$$

where $H_+ = (H_{rr} + H_{\theta\theta})/2$ and $H_- = (H_{rr} - H_{\theta\theta})/2$ are combinations of the zeroth-order (axisymmetric) pressure Hessian.

3.1 Angular structure

Substituting the explicit expressions for ΔS_{12} and ΔS_{32}, we find:

$$Q^{(1)} = A\cos\varphi + B\sin\varphi$$

where A and B are radial functions depending on the Hessian components and eigenvalue gaps. At $\rho = \sigma$ with $\mathrm{Re} = 100$, $s = 10$, $\kappa = 0.01$:

$$Q^{(1)}(\rho = \sigma, \varphi) = 2.505\cos\varphi + 1.510\sin\varphi = 2.925\cos(\varphi - 31°)$$

This is a pure $m = 1$ (dipolar) angular mode. **Theorem (Dipolar Oscillation).** *For a Burgers vortex with uniform curvature κ, the first-order pressure-Hessian observable $Q^{(1)}$ has $m = 1$ angular dependence. Its enstrophy-weighted cross-sectional average vanishes:*

$$\langle Q^{(1)} \rangle_\omega \equiv \frac{\int Q^{(1)}\ |\omega|^2\, dA}{\int |\omega|^2\, dA} = 0$$

Proof. The angular factor of $Q^{(1)}$ is $A\cos\varphi + B\sin\varphi$, which is a pure $m = 1$ mode. The enstrophy $|\omega|^2 = \omega_0^2 e^{-2\rho^2/\sigma^2}$ is axisymmetric ($m = 0$). The integral of an $m = 1$ function against an $m = 0$ weight over $[0, 2\pi]$ vanishes. □ The theorem has a deeper structural content than the mere vanishing of an integral. The first mode unlocked by symmetry breaking ($m = 1$, from $O(\kappa)$ curvature) is *orthogonal to the weighting measure* ($m = 0$ enstrophy). This is not accidental: the curvature perturbation is a translation-like deformation (displacing the axis), which generates a

φ	$Q^{(1)}$	**Sign**
0 (outward normal)	+2.504	positive
$\pi/4$	+2.838	maximum
$\pi/2$ (binormal)	+1.510	positive
$3\pi/4$	−0.703	negative
π (inward normal)	−2.504	negative
$5\pi/4$	−2.838	minimum
$3\pi/2$	−1.510	negative
$7\pi/4$	+0.703	positive

Table 1: First-order Q at $\rho = \sigma$, $s = 10$, $\kappa = 0.01$, Re = 100. The cross-section average is $-2.0 \times 10^{-15} \approx 0$.

dipole, while the weighting measure retains the unperturbed azimuthal symmetry. Any mechanism that produces net $\langle Q \rangle \neq 0$ must therefore either (a) excite $m = 0$ or $m = 2$ modes in Q directly, or (b) break the axisymmetry of $|\omega|^2$ so that the weight itself contains compatible harmonics. This is why single-tube curvature fails at first order and why tube-tube interactions — which generically break the $m = 0$ symmetry of the vorticity weight — are the natural next candidate. The trajectory average also vanishes at leading order. A fluid element at radius ρ_0 orbiting with angular velocity $\Omega_0 = v_\theta(\rho_0)/\rho_0$ samples $Q(t) = [A\cos(\Omega_0 t + \varphi_0) + B\sin(\Omega_0 t + \varphi_0)] \times \gamma s(t)\kappa/2$. Since $\gamma \ll \Omega_0$ at high Reynolds number, the oscillation averages out over each circulation period.

4 Source Perturbation and Pressure Response

The curvature also modifies the diagonal strain component e_{ss} at $O(\kappa)$:

$$\Delta e_{ss} = -(\gamma\rho/2)\kappa\cos\varphi - \kappa v_\theta \sin\varphi$$

The dominant term at high Reynolds number is $-\kappa v_\theta \sin\varphi$ (since $v_\theta \sim \omega_0\sigma \gg \gamma\sigma$). This changes the source $g = |S|^2 - 1/2\ |\omega|^2$ at $O(\kappa)$:

$$\Delta g = \kappa[g_c(\rho)\cos\varphi + g_s(\rho)\sin\varphi]$$

where:

$$g_c(\rho) = \rho\left(\omega_0^2 e^{-2\rho^2/\sigma^2} - \gamma^2\right) \approx \rho\omega_0^2 e^{-2\eta}$$

$$g_s(\rho) = -2\gamma v_\theta(\rho)$$

The $\cos\varphi$ source g_c dominates by a factor $\omega_0^2/\gamma^2 = \mathrm{Re}^2$. It arises from the metric correction to the vorticity: on the outside of the bend, the tube is metrically compressed, weakening the vorticity. Each mode satisfies an $m = 1$ radial Poisson ODE:

$$p_c'' + p_c'/\rho - p_c/\rho^2 = g_c(\rho)$$

with regularity at $\rho = 0$ and decay at $\rho \to \infty$. We solve this via ode45 integration with boundary matching (subtracting the growing homogeneous solution $C\rho$ to enforce decay).

4.1 Hessian components at $\rho = \sigma$

The physical solution $p_c(\rho)$ (after boundary matching) yields the Hessian components at $\rho = \sigma$:

$$A_c \equiv p_c'' = 633, \quad B_c \equiv p_c'/\rho - p_c/\rho^2 = 742$$

These are large because the source $g_c \approx \rho\omega_0^2 e^{-2\eta}$ is driven by the peak vorticity. The trace-like combination $A_c + B_c = 1375$ and anisotropy $A_c - B_c = -109$ control the different contributions to Q. The pressure perturbation is an in-plane (z-independent) function $\Delta p = \kappa p_c(\rho)\cos\varphi$. Its Hessian has no H_{xz} or H_{yz} components (since $\partial^2 \Delta p/\partial z \partial x = 0$). Therefore term (A) of the perturbation expansion — $e_2^0 \cdot \Delta H^{(1)} \cdot e_1^0 = \hat{z} \cdot \Delta H \cdot e_1^0$ — vanishes. The only first-order contribution to Q comes from the eigenframe perturbation (terms $B + C$), as computed in Section 3. **Robustness.** The sign of $\langle Q \rangle^{(2)}$ is controlled by two structural features: (i) $A_c + 3B_c > 0$, which holds whenever the source $g_c(\rho)$ is dominated by vorticity rather than strain ($\omega_0 \gg \gamma$, i.e., $\mathrm{Re} \gg 1$), since the Poisson solution inherits the positivity of the enstrophy source; and (ii) $\lambda_2 - \lambda_1 < 0$, which is guaranteed by the ordering convention $\lambda_1 \geq \lambda_2$. At $\mathrm{Re} = 50$, the values shift to $A_c = 127$, $B_c = 156$, giving $A_c + 3B_c = 595 > 0$ and $\langle Q \rangle^{(2)} > 0$ with the same sign. The anti-depletion result is robust across the high-Reynolds-number regime relevant to potential blow-up and does not depend on the specific choice $\mathrm{Re} = 100$.

5 Second-Order Net Q: The Anti-Depletion Result

The first nonzero cross-section average of Q appears at $O(\kappa^2)$, from the product of two $m = 1$ quantities:

- The eigenframe perturbation $e_2^{(1)}$ from $\Delta S_{xz} = -\gamma s \kappa/2$ (Section 3)
- The Hessian perturbation $\Delta H^{(1)}$ from the pressure response $\kappa p_c(\rho)\cos\varphi$ (Section 4)

The $m = 0$ component of this $m = 1 \times m = 1$ product gives a nonzero cross-section average. Explicitly, term (b) of the second-order expansion is:

$$Q_b^{(2)} = e_2^{(1)} \cdot \Delta H^{(1)} \cdot e_1^0$$

where $e_2^{(1)} = [\Delta S_{12}/(\lambda_2 - \lambda_1)]e_1^0 + [\Delta S_{32}/(\lambda_2 - \lambda_3)]e_3^0$. After computing the azimuthal average (using $\langle \cos^2\varphi \rangle = 1/2$, $\langle \sin^2\varphi \rangle = 1/2$, $\langle \cos\varphi \sin\varphi \rangle = 0$):

$$\langle Q_b^{(2)} \rangle_\varphi = -\frac{\gamma s \kappa^2}{8\sqrt{2}} \left[\frac{A_c + 3B_c}{\lambda_2 - \lambda_1} + \frac{A_c - B_c}{\lambda_2 - \lambda_3} \right]$$

5.1 Evaluation and sign determination

With the computed values at $\rho = \sigma$, $\mathrm{Re} = 100$:

$$\frac{A_c + 3B_c}{\lambda_2 - \lambda_1} = \frac{2858}{-11.71} = -244.0$$

$$\frac{A_c - B_c}{\lambda_2 - \lambda_3} = \frac{-109}{14.71} = -7.4$$

$$\text{Sum} = -251.4$$

$$\text{Prefactor:} \quad -\frac{\gamma s \kappa^2}{8\sqrt{2}} = -\frac{1 \cdot 10 \cdot 10^{-4}}{8\sqrt{2}} = -8.84 \times 10^{-5}$$

$$\langle Q \rangle^{(2)} = (-8.84 \times 10^{-5}) \times (-251.4) = +\mathbf{0.022}$$

The sign is *positive.* **Theorem (Anti-Depletion).** *For a Burgers vortex with uniform curvature κ at high Reynolds number, the leading-order cross-section-averaged Q is positive:*

$$\langle Q \rangle^{(2)} > 0$$

Hence single-tube curvature is anti-depleting: the pressure-Hessian response to geometric bending promotes alignment with the most stretching strain direction rather than opposing it. The positivity arises from two factors:

1. The dominant Hessian combination $A_c + 3B_c = 2858 > 0$: the curvature-induced pressure perturbation has strong positive gradient structure in the core.
2. The eigenvalue gap $\lambda_2 - \lambda_1 = -11.71 < 0$: the intermediate eigenvalue lies below the most stretching one. The division by this negative gap reverses the expected sign.

6 The Vortex Ring: An Exact Vanishing Case

The bent Burgers tube has two ingredients: curvature κ and axial stretching $v_s = \gamma s$. A natural question is whether curvature *alone* suffices to produce $Q \neq 0$. The vortex ring — a closed tube with curvature $\kappa = 1/R$ but no external stretching — is the canonical test case. **Theorem (Ring Vanishing).** *For a non-swirling axisymmetric vortex ring with purely toroidal vorticity, $Q = 0$ exactly at all orders in the curvature $\kappa = 1/R$. Proof.* We work in Frenet-frame toroidal coordinates (ρ, φ, s) with the same metric $h_s = 1 + \kappa\rho\cos\varphi$ as the bent tube. A standard (non-swirling) ring has $v_s = 0$ (no toroidal velocity component) and toroidal symmetry ($\partial/\partial s = 0$ for all flow variables). (1) *Off-diagonal strain vanishes:* The strain components coupling axial to in-plane directions are:

$$e_{\rho s} = \frac{H}{2}\frac{\partial}{\partial \rho}\left(\frac{v_s}{H}\right) + \frac{1}{2H}\frac{\partial v_\rho}{\partial s} = 0 + 0 = 0$$

$$e_{\varphi s} = \frac{H}{2\rho}\frac{\partial}{\partial \varphi}\left(\frac{v_s}{H}\right) + \frac{\rho}{2H}\frac{\partial}{\partial s}\left(\frac{v_\varphi}{\rho}\right) = 0 + 0 = 0$$

(2) *Block-diagonal strain:* With $e_{\rho s} = e_{\varphi s} = 0$, the strain tensor decomposes as $(\rho, \varphi) \oplus (s)$. The in-plane block has eigenvectors e_1, e_3 in the $(\hat{\rho}, \hat{\varphi})$ plane; the axial direction $\hat{e}_s$ is an eigenvector with eigenvalue $\lambda_2 = e_{ss}$. This is exact — not a perturbative statement — because the block-diagonal structure holds at all orders in κ. (3) *Axisymmetric pressure:* The pressure depends only on (ρ, φ), not on s, so $\partial p/\partial s = 0$. All covariant derivatives of ∇p in the s-direction also vanish (the connection coefficients $\nabla_{\hat{e}_s}\hat{e}_\rho$ and $\nabla_{\hat{e}_s}\hat{e}_\varphi$ are proportional to $\partial p/\partial s = 0$). Hence $H_{s\rho} = H_{s\varphi} = 0$. (4) *Therefore:* $Q = e_2 \cdot H_{\mathrm{tf}} \cdot e_1 = \hat{e}_s \cdot H_{\mathrm{tf}} \cdot e_1 = 0$ because H_{tf} has no s-cross components and e_1 has no s-component. $\square$ The ring does have a nonzero diagonal strain at $O(\kappa)$:

$$e_{ss} = -\kappa v_\theta(\rho)\sin\varphi/H$$

which is $m = 1$ and creates an $m = 1$ pressure perturbation and eigenvalue modulation. But this does not *tilt* the eigenframe — it only changes λ_2. The eigenframe tilt that drives $Q \neq 0$ requires off-diagonal strain $e_{\rho s}$ or $e_{\varphi s}$, which are identically zero when $v_s = 0$.

6.1 The essential ingredient for $Q \neq 0$

Comparing the ring ($Q = 0$ exactly) with the bent Burgers tube ($Q \neq 0$ at $O(\kappa)$) reveals the essential ingredient:

$$\Delta S_{xz} = -v_s\kappa/2$$

The off-diagonal strain arises from the *product* of curvature κ and axial velocity v_s. Neither alone suffices:

- Curvature without axial flow (ring): $v_s = 0 \to \Delta S_{xz} = 0 \to Q = 0$.
- Axial flow without curvature (straight Burgers tube): $\kappa = 0 \to \Delta S_{xz} = 0 \to Q = 0$.
- Both together (bent Burgers tube): $v_s = \gamma s$, $\kappa > 0 \to \Delta S_{xz} = -\gamma s\kappa/2 \neq 0 \to Q \neq 0$.

This is a structural decomposition of what breaks the vanishing theorems: *curvature alone is necessary but not sufficient; the mechanism requires an axial velocity gradient along a curved axis.* The physical picture is clear. In the ring, the flow circulates in poloidal planes (swirl); there is no velocity component along the ring axis. The strain eigenframe is locked to the plane-plus-axis structure at all orders. In the bent Burgers tube, fluid moves *along* the curved axis ($v_s = \gamma s$), and the Frenet-frame metric creates a differential velocity between the inside and outside of the bend, tilting the eigenframe. The persistence of vortex rings reflects not a depleting mechanism but a symmetry-protected state in which the pressure-Hessian observable vanishes identically. Their eventual breakdown — the decoherence into turbulence observed in experiments — corresponds to the loss of this symmetry and the onset of interaction-driven dynamics, where the axisymmetric vorticity weight $|\omega|^2$ is broken by external perturbations and the orthogonality argument of the Dipolar Oscillation Theorem no longer applies.

7 Physical Interpretation

The anti-depletion result has a clear physical interpretation. Consider a vortex tube bent with curvature κ in the x-z plane. The metric correction $h_s = 1 + \kappa\rho\cos\varphi$ compresses the tube slightly on the outside of the bend ($\varphi = 0$) and expands it on the inside ($\varphi = \pi$). This creates an asymmetry in the vorticity distribution at $O(\kappa)$. The pressure responds to balance this asymmetry: on the side with metrically compressed (stronger) vorticity, the centrifugal force is slightly higher, creating a pressure maximum. On the other side, the pressure is lower. This dipolar pressure perturbation creates a Hessian with $m = 1$ angular dependence. Simultaneously, the curvature creates an off-diagonal strain $\Delta S_{xz} = -\gamma s\kappa/2$ that tilts the strain eigenframe. The tilt direction is *correlated* with the pressure response: both arise from the same curvature perturbation and share the same $m = 1$ symmetry. The $O(\kappa^2)$ cross-term between the eigenframe tilt and the Hessian perturbation has a definite sign because the tilt and the pressure response are in phase. The sign is positive, meaning that the combined effect pushes vorticity *toward* the most stretching direction. Physically:

1. Curvature creates a pressure maximum on the outside of the bend.
2. This pressure maximum acts to *maintain* the tube's alignment with the stretching direction.
3. The tube effectively resists deformation through its own pressure response.

This is the *self-protection* mechanism of vortex tubes. It explains the observed persistence of coherent vortex tubes in turbulent flows: if single-tube curvature depleted alignment, tubes would self-destruct. Instead, the pressure Hessian's primary role on an isolated tube is structural maintenance.

8 Two-Tube Interaction: The Tidal Gradient Mechanism

The preceding sections established that single-tube geometries either vanish ($Q = 0$) or are anti-depleting ($\langle Q \rangle > 0$). We now compute Q for the simplest interacting configuration that breaks z -translational symmetry. **Setup.** Tube A is a Burgers vortex along $\hat{z}$ at the origin. Tube B is a line vortex along $\hat{x}$ at position $(0, d, 0)$ with circulation Γ_B. Perpendicular tubes are essential: two *parallel* tubes (both along $\hat{z}$) preserve z-translational symmetry, so $Q = 0$ by the z-Translation Vanishing Theorem [2]. Perpendicular tubes break this symmetry because tube B's strain at A varies with z: the distance from A's axis to B is $\sqrt{d^2 + z^2}$, so $S_{yz} \propto 1/(d^2 + z^2)$. **Selection rule.** The natural expectation is that the $m = 1$ curvature modes (Section 3) couple with the $m = 2$ tidal modes from the external tube. However, the product of $m = 1$ and $m = 2$ gives $m = 1$ and $m = 3$ only — never $m = 0$:

$$\int_0^{2\pi} \cos\varphi \cos 2\varphi \, d\varphi = 0$$

This is a robust Fourier selection rule that prevents direct curvature-tidal coupling at any perturbative order. The mechanism for nonzero $\langle Q \rangle$ must come from elsewhere.

8.1 The tidal gradient mechanism

Tube B creates an off-diagonal strain S_{yz} at tube A's cross-section. At $z = 0$:

$$S_{yz}(y) = -\frac{\Gamma_B}{2\pi(d - y)^2} \approx \varepsilon_0 + \varepsilon_1 y$$

where $\varepsilon_0 = -\Gamma_B/(2\pi d^2)$ is the uniform strain and $\varepsilon_1 = -\Gamma_B/(\pi d^3)$ is the *tidal gradient*. The uniform part ε_0 creates $m = 1$ eigenbasis projections — the same dipolar structure as the bent tube — and averages to zero by the Dipolar Oscillation Theorem. The gradient part $\varepsilon_1 y = \varepsilon_1 \rho \sin\varphi$ has $m = 1$ angular dependence in Cartesian coordinates. But when projected onto the *cylindrical strain eigenbasis*, which is rotated by $\pi/4$ relative to the Cartesian basis, it generates an $m = 0$ component:

$$\Delta S_{12} = S_{yz} \frac{\sin\varphi + \cos\varphi}{\sqrt{2}}$$

Substituting $S_{yz} = \varepsilon_1 \rho \sin\varphi$ for the gradient part:

$$\Delta S_{12}^{\text{grad}} = \frac{\varepsilon_1 \rho}{\sqrt{2}}(\sin^2\varphi + \sin\varphi\cos\varphi) = \frac{\varepsilon_1\rho}{2\sqrt{2}}[1 + \sin 2\varphi - \cos 2\varphi]$$

The $m = 0$ component is $\varepsilon_1\rho/\left(2\sqrt{2}\right)$. This is the key insight: the eigenbasis rotation converts Cartesian $m = 1$ into cylindrical $m = 0$. The cross-section-averaged Q is therefore:

$$\langle Q\rangle_\varphi(\rho) = \frac{\varepsilon_1\rho}{2\sqrt{2}}F(\rho)$$

where $F(\rho) = \frac{H_+ - H_{zz}}{\lambda_2-\lambda_1} + \frac{H_-}{\lambda_2-\lambda_3}$ is the *interaction kernel* — the same radial function that controls the single-tube eigenframe tilt.

8.2 Sign determination

The interaction kernel $F(\rho)$ is *uniformly negative* throughout the vortex core:

$$F(0.3\sigma) = -3421, \quad F(\sigma) = -70.8, \quad F(2\sigma) = -8.8, \quad F(3\sigma) = -1.4$$

The negativity arises because $\lambda_2 - \lambda_1 < 0$ in the core (the intermediate eigenvalue $\lambda_2 = \gamma$ lies below the most stretching $\lambda_1 = -\gamma/2 + |S_{r\theta}|$ at high Reynolds number). Since $F(\rho) < 0$ uniformly, the sign of $\langle Q\rangle$ is determined by $\varepsilon_1 = -\Gamma_B/(\pi d^3)$:

- If $\varepsilon_1 > 0$ (stretching-enhancing, $\Gamma_B < 0$): $\langle Q\rangle < 0$ — **depleting**.
- If $\varepsilon_1 < 0$ (stretching-opposing, $\Gamma_B > 0$): $\langle Q\rangle > 0$ — anti-depleting.

The enstrophy-weighted cross-section average obeys the scaling law

$$\langle Q\rangle_\omega = C\,\gamma^2\mathrm{Re}^2(\sigma/d)^3$$

where $C \approx -0.55$ is a negative dimensionless constant determined entirely by the Burgers vortex radial profile, independent of Re, γ, σ, or d individually. At the representative values $\mathrm{Re} = 100$, $d = 10\sigma$ (with $\gamma = \sigma = 1$), this gives $\langle Q\rangle_\omega = -5.52$ for the stretching-enhancing case ($\Gamma_B = -\Gamma_A$). The Re^2 scaling is physically significant: the interaction-induced depletion *strengthens* toward the high-Reynolds-number regime relevant to potential blow-up. The $(\sigma/d)^3$ factor reflects the tidal-gradient decay with separation. At the same representative parameters, the interaction dominates the single-tube self-protection $\langle Q\rangle^{(2)} = +0.022$ by a factor of ≈ 250. This ratio depends on the comparison point — the separation d/σ for the interaction, the curvature $\kappa\sigma$ and station s/σ for the self-protection — and should not be read as a universal constant. **Theorem (Interaction Depletion).** *For two perpendicular vortex tubes with stretching-enhancing circulation, the enstrophy-weighted cross-section average of the pressure-Hessian observable is negative:*

$$\langle Q\rangle_\omega < 0$$

The dominant interaction effect is depleting and overwhelms the single-tube anti-depletion. **Physical interpretation.** The sign depends on whether the interaction enhances or opposes tube A's axial stretching. In turbulence, vortex tubes form precisely because the surrounding velocity field stretches them; the dominant pairwise interactions are therefore stretching-enhancing. The tidal gradient mechanism ensures that these interactions produce $Q < 0$, providing the first constructive derivation of the depleting sign from first principles. The correlation between stretching enhancement and depletion is not a coincidence — it is a structural consequence of the interaction kernel

$F(\rho) < 0$, which is negative whenever the in-plane shear exceeds the axial strain rate ($\lambda_1 > \lambda_2$), a condition satisfied throughout the core at high Reynolds number.

9 Implications for the Regularity Problem

The results of this paper, combined with those of [1] and [2], establish a hierarchy of vanishing and sign results for the pressure-Hessian observable Q:

1. **Straight tube** ($\kappa = 0$, z-translationally symmetric): $Q = 0$ identically [2].
2. **Elliptically deformed straight tube** (still z-symmetric): $Q = 0$ by the z-Translation Vanishing Theorem [2].
3. **Vortex ring** ($\kappa = 1/R > 0$, but no axial flow): $Q = 0$ exactly at all orders (Ring Vanishing Theorem, Section 6).
4. **Bent Burgers tube** ($\kappa > 0$ and $v_s = \gamma s \neq 0$, first genuine escape): (a) $Q \neq 0$ at $O(\kappa)$, but purely dipolar ($m = 1$), so $\langle Q \rangle = 0$. (b) $\langle Q \rangle^{(2)} > 0$ at $O(\kappa^2)$: anti-depleting.

This progression systematically eliminates candidate mechanisms:

- Cross-sectional deformation is eliminated by the z-Translation Vanishing Theorem.
- Curvature without axial flow is eliminated by the Ring Vanishing Theorem.
- Curvature with axial flow is eliminated by the Anti-Depletion Theorem.

What remains? The depletion mechanism must break the axisymmetry of the *vorticity distribution itself*, not merely the geometry of a single tube. Three candidate mechanisms satisfy this requirement: **(A) Tube-tube interactions.** When two vortex tubes approach each other, each tube's vorticity creates a strain field that acts on the other. This external strain does not share the single tube's axisymmetry, so the $m = 1$ field of Q is weighted against a non-axisymmetric $|\omega|^2$, producing a nonzero average. **(B) Non-axisymmetric background strain.** In turbulence, the background strain that stretches a tube is generically anisotropic in the cross-sectional plane. This breaks the azimuthal symmetry and creates an $m = 2$ or higher correction to $|\omega|^2$, which can couple with the $m = 1$ dipolar Q to produce a net bias. **(C) Curvature variation.** If the curvature κ varies along the tube ($d\kappa/ds \neq 0$), the local s-reflection symmetry is broken, creating additional contributions that do not cancel pairwise.

9.1 The pairwise interaction kernel

The two-tube computation (Section 8) reveals that the mechanism for $\langle Q \rangle < 0$ is not tidal deformation of the vorticity weight, but the *tidal gradient* of the off-diagonal strain acting through the eigenbasis projection. This mechanism: (a) Requires perpendicular (not parallel) tubes to break z-symmetry. (b) Bypasses the $m = 1 \times m = 2 \to 0$ selection rule by converting Cartesian $m = 1$ into cylindrical $m = 0$ through the $\pi/4$ eigenbasis rotation. (c) Produces a sign controlled by the interaction kernel $F(\rho) < 0$, which is negative throughout the core for any vortex tube whose in-plane shear exceeds the axial strain rate — a condition equivalent to $\mathrm{Re} \gg 1$ and satisfied by all tubes in the regime relevant to potential blow-up. (d) Correlates the depleting sign with stretching enhancement: the pairwise interactions that sustain vortex tubes in turbulence are precisely those that produce $Q < 0$. The hierarchy of effects is now clear: single-tube self-protection is overwhelmed by pairwise interaction depletion. Both effects scale as Re^2, so their ratio is independent of

Reynolds number. The interaction scales as $(\sigma/d)^3$ while the self-protection scales as $(\kappa\sigma)^2$; at representative parameters ($d = 10\sigma$, $\kappa\sigma = 0.01$) the ratio is ≈ 250, showing that the cooperative depleting mechanism dominates at any moderate separation.

9.2 Interaction inevitability

A natural question is whether the interaction mechanism is merely a possibility or a structural necessity of blow-up. We argue, via two Z3-verified lemmas and a self-consistency argument, that the blow-up scenario *forces* the system into the perturbative regime where the tidal gradient mechanism applies — and that the depletion strengthens as blow-up intensifies. **Tidal gradient locality (Z3-verified).** The $m = 0$ component of Q that survives cross-section averaging arises solely from the tidal gradient $\varepsilon_1 \sim \Gamma/d^3$ — not the uniform strain $\varepsilon_0 \sim \Gamma/d^2$, which produces only dipolar ($m = 1$) contributions that average to zero. Since ε_1 decays as $1/d^3$, the sum over all tubes converges rapidly: $\sum_{k=2}^{\infty} 1/k^3 = \zeta(3) - 1 \approx 0.202$, so the nearest tube contributes at least 83% of the total tidal gradient (Z3 proves the bound $\varepsilon_{1,\text{near}} \geq 0.8\, \varepsilon_{1,\text{total}}$). Distant vorticity contributes background strain but *not* averaged Q. **Self-consistent separation scaling.** When tube A's background strain γ is provided by a nearby tube B at distance d, the self-consistency condition $\gamma = \Gamma_B/(2\pi d^2)$ combined with $\sigma^2 = 2\nu/\gamma$ and $\Gamma = 2\pi\nu\,\text{Re}$ yields:

$$d/\sigma = \sqrt{\text{Re}\,/2}$$

Three consequences follow, all Z3-verified: (a) **Perturbative validity strengthens.** The expansion parameter of the tidal gradient analysis is $\varepsilon_1/(\lambda_1 - \lambda_2) \sim \text{Re}^{-3/2}$, which *shrinks* as Re grows. At $\text{Re} = 100$ this ratio is 0.025; at $\text{Re} = 10^4$ it is 10^{-4}. The perturbative regime becomes *exact* in the blow-up limit. (b) **Tube separation grows.** $d/\sigma = \sqrt{\text{Re}\,/2} \to \infty$ as $\text{Re} \to \infty$. The tubes are well-separated relative to their cores, precisely the regime where the Biot-Savart tidal expansion is most accurate. (c) **Depletion strengthens.** The interaction Q under self-consistent scaling becomes $\langle Q \rangle_\omega = C\,\gamma^2\sqrt{\text{Re}}$ with $C < 0$ (Z3-verified). The depleting effect *grows* as $\sqrt{\text{Re}}$ toward blow-up — the mechanism does not fade but sharpens. **The self-undermining property.** Combining these observations: a blow-up scenario requires high vorticity ($\omega_0 \to \infty$) sustained by external stretching. In the vortex-tube regime ($\text{Re} \gg 1$), this stretching comes from nearby tubes via Biot-Savart, with self-consistent separation $d/\sigma = \sqrt{\text{Re}\,/2}$. The same interaction that sustains the tube (stretching-enhancing) produces $Q < 0$ with growing magnitude. The blow-up scenario is *self-undermining*: the mechanism that would drive it simultaneously produces the depletion that opposes it. **Regime assumption.** The above argument assumes vorticity concentrates in Burgers-type tubes with $\text{Re} \gg 1$ — a well-supported but unproven assumption for blow-up configurations. Whether all admissible blow-up scenarios must pass through tube-dominated configurations remains the principal open question.

10 Synthesis

The complete reduction chain is:

1. Scalar Sobolev methods cannot decide the problem [1].
2. The missing mechanism localizes to $Q = e_2 \cdot H_{\text{tf}} e_1$ [2].

3. $Q = 0$ for all z-translationally symmetric flows [2].
4. $Q = 0$ exactly for the vortex ring — curvature alone is not sufficient (this paper).
5. $Q \neq 0$ requires curvature *plus* axial flow; the bent Burgers tube is the first escape (this paper).
6. The first nonzero mode is dipolar ($m = 1$), averaging to zero (Dipolar Oscillation Theorem, this paper).
7. The first nonzero mean is $\langle Q \rangle^{(2)} > 0$: anti-depleting (this paper).
8. The $m = 1 \times m = 2$ coupling vanishes by Fourier selection rules (this paper).
9. The *tidal gradient* mechanism from perpendicular tube interaction creates an $m = 0$ eigenbasis projection that survives averaging (this paper).
10. The resulting $\langle Q \rangle_\omega = C\,\gamma^2 \mathrm{Re}^2 (\sigma/d)^3 < 0$ ($C \approx -0.55$): the first constructive derivation of the depleting sign, with Re^2 scaling toward blow-up (this paper).
11. Tidal gradient locality: the $m = 0$ mechanism requires *nearby* tubes ($\varepsilon_1 \sim 1/d^3$ convergence), so averaged Q is controlled by the nearest tube (Z3-verified, this paper).
12. Self-consistent blow-up scaling: $d/\sigma = \sqrt{\mathrm{Re}\,/2}$, perturbation parameter $\sim \mathrm{Re}^{-3/2} \to 0$, depletion $\sim \sqrt{\mathrm{Re}} \to \infty$ (Z3-verified, this paper).
13. The blow-up scenario is *self-undermining*: the stretching-enhancing interactions that would sustain blow-up are precisely those that produce $Q < 0$ with growing magnitude (this paper).

The transition from single-tube vanishing (steps 3–7) to interaction depletion (steps 9–10) is not merely a change of target but a structural necessity. The single-tube vanishing is due to symmetry orthogonality: the first unlocked mode ($m = 1$) is incompatible with the axisymmetric weighting measure ($m = 0$). The interaction breaks this orthogonality through the tidal gradient — not by modifying the weight, but by creating an $m = 0$ component of the Q field itself through the eigenbasis rotation. The self-protection of individual tubes (positive $\langle Q \rangle$) and the interaction depletion (negative $\langle Q \rangle$) coexist: the former preserves coherent tubes while the latter drives enstrophy growth when tubes interact. Since both effects scale as Re^2, their ratio is Reynolds-number-independent; the interaction dominates the self-protection at any moderate separation (e.g., by a factor $\approx$ 250 at $d = 10\sigma$, $\kappa\sigma = 0.01$). The inevitability argument (steps 11–13) tightens the narrative from *mechanism identification* to *structural necessity*: the depletion is not merely possible but forced by the blow-up dynamics, under the tube-structure assumption.

11 Conclusion

We have computed the pressure-Hessian observable $Q = e_2 \cdot H_{\mathrm{tf}} e_1$ for curved vortex geometries and interacting tubes, establishing six results:

1. **Ring Vanishing Theorem:** A non-swirling vortex ring has $Q = 0$ exactly at all orders in $\kappa = 1/R$. Curvature alone is necessary but not sufficient for $Q \neq 0$.
2. **First nonzero Q:** The bent Burgers tube ($\kappa > 0$, $v_s = \gamma s \neq 0$) creates $Q \neq 0$ — the first model in our chain to escape the vanishing classes.
3. **Dipolar Oscillation Theorem:** The first-order Q is a pure $m = 1$ mode whose cross-section average vanishes.
4. **Anti-Depletion Theorem:** The first nonzero mean is $\langle Q \rangle^{(2)} = +0.022 > 0$. Single-tube curvature is anti-depleting.

5. **Interaction Depletion Theorem:** For perpendicular vortex tubes with stretching-enhancing circulation, the tidal gradient mechanism produces $\langle Q \rangle_\omega = C\gamma^2 \mathrm{Re}^2(\sigma/d)^3$ with $C \approx -0.55$ — a robust depleting sign that scales as Re^2 toward blow-up and overwhelms the single-tube self-protection at any moderate separation.
6. **Interaction Inevitability:** Under the tube-structure assumption, the blow-up scenario is self-undermining. Self-consistent scaling gives $d/\sigma = \sqrt{\mathrm{Re}\,/2}$, forcing the system into the perturbative regime (expansion parameter $\sim \mathrm{Re}^{-3/2} \to 0$) where the tidal gradient mechanism applies with *growing* magnitude ($Q \sim \sqrt{\mathrm{Re}} \to \infty$). Z3 verifies the complete chain: blow-up + tube structure $\Rightarrow Q < 0$.

The complete narrative is:

Curvature unlocks the observable; **interaction determines the sign**.

Single-tube curvature breaks the vanishing theorems (Ring $\to$ Bent Tube $\to$ Dipolar Oscillation $\to$ Anti-Depletion) but produces the wrong sign. The correct depleting sign emerges from the *tidal gradient* of the off-diagonal strain in vortex-vortex interaction — a mechanism that converts Cartesian $m = 1$ strain variation into cylindrical $m = 0$ through the $\pi/4$ eigenbasis rotation. The sign is controlled by the interaction kernel $F(\rho) < 0$, which is negative throughout the core whenever the in-plane shear exceeds the axial strain rate ($\mathrm{Re} \gg 1$), and correlates with stretching enhancement. The inevitability argument (Result 6) strengthens this from mechanism identification to structural necessity: the self-consistent blow-up dynamics *forces* the system into the regime where the depleting mechanism applies and sharpens. Within the tube-structure regime, blow-up is self-undermining — the stretching-enhancing interactions that would sustain it are precisely those that produce $Q < 0$. The principal remaining question is whether all admissible blow-up scenarios must pass through tube-dominated configurations. The tube-structure assumption is supported by DNS evidence [4] and the Burgers vortex as the canonical stretched-vortex solution, but it is not a theorem. The reduction from the abstract regularity question — does H^1 blow up? — to a computable pairwise kernel — is $F(\rho)$ negative in the core? — to a self-consistency argument — does blow-up force the depleting regime? — represents a progressive structural narrowing of the problem. Whether this chain closes into a proof depends on formalizing the tube-structure assumption or, equivalently, showing that vorticity concentration is inevitable at blow-up.

11 References

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