The Half-Derivative Gap: Machine-Verified Structural Diagnosis of Navier-Stokes Smoothness

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Abstract

We axiomatize the Sobolev norm hierarchy of the 3D incompressible Navier-Stokes equations and use the Z3 SMT solver to test whether blow-up scenarios are consistent with the resulting scalar constraint systems. Twenty-six satisfiability tests cover interpolation inequalities, Kato-Ponce and fractional Gronwall bounds, Millennium Problem boundary conditions ($\mathbb{T}^3$, $\nabla \cdot v = 0$), all four conservation laws including the Leray energy inequality, and the Cauchy-Green deformation tensor. Every test at the standard viscous exponent $\alpha = 1$ returns SAT: blow-up is not excluded. Two controls return UNSAT: the 2D case ($S = 0$) and the Lions hyperviscous threshold ($\alpha = 5/4$), isolating the half-derivative gap $\mathrm{gap}(d) = d/2 - 1$ as the organizing invariant.

We then test non-scalar axiom candidates of the form $S^2 \leq \Omega^a P^b$. The gap closes if and only if the exponent sum $a + b \leq 2$; at this threshold, the basin of stability collapses onto the entire state space (global basin collapse). The standard bound has $a + b = 4$; Kato-Ponce already reduces this to $a + b = 3$. Within this framework's exponent bookkeeping, the remaining reduction is exactly one derivative order, independent of how the exponents are distributed. The Millennium Problem is undecidable within the scalar Sobolev framework: the full Navier-Stokes equations may implicitly enforce a geometric depletion of stretching, but that enforcement is invisible after projection to scalar norm inequalities.

Keywords: Navier-Stokes equations, Millennium Problem, formal verification, SMT solver, Sobolev spaces, fractional derivatives, Riemann-Liouville, Kato-Ponce, Lions theorem, structural diagnosis

1 Introduction

The Navier-Stokes existence and smoothness problem, one of the seven Clay Mathematics Institute Millennium Prize Problems, asks whether smooth, divergence-free initial data on $\mathbb{R}^3$ or $\mathbb{T}^3$ necessarily yield a smooth solution for all time [1]. Despite a century of progress in harmonic analysis, functional analysis, and PDE theory, the problem remains open.

The difficulty is well-characterized: in three dimensions, the vortex stretching term $\omega \cdot \nabla v$ can amplify vorticity, potentially driving $\|\omega\|_{L^\infty}$ to infinity in finite time. The viscous dissipation $-\nu \Delta v$ opposes this, but the two effects operate at different derivative orders: stretching involves ∇v (one derivative), while dissipation involves Δv (two derivatives). Whether dissipation always dominates stretching in 3D is unknown.

This paper takes a different approach from both the analytical and computational traditions. Rather than attempting a proof or a numerical simulation, we treat the problem as a *structural diagnosis*: we axiomatize the known constraints on the Sobolev norm dynamics and use an SMT (Satisfiability Modulo Theories) solver to determine which axiom systems are consistent with blow-up and which exclude it.

The methodology is inspired by the Spectral Comb investigation of the Riemann Hypothesis [2], where Z3 satisfiability tests identified the antisymmetric structure as the key architectural element. Here, we apply the same formal verification framework to the Navier-Stokes equations, with three goals: (i) determine whether blow-up is consistent with a broad family of axiomatized scalar constraints; (ii) identify which additional axioms would make blow-up inconsistent (UNSAT); and (iii) characterize the *type* of structure that is missing from the scalar framework.

All computations are performed in the Kleis formal verification language [4], with the Z3 SMT solver providing satisfiability verdicts. Each test completes in under two seconds. The full test suite is available in the `theories/` directory of the Kleis repository.

1.1 Methodological remark

This investigation follows a diagnostic methodology: rather than attacking the Millennium Problem directly, we axiomatize the known scalar machinery and use machine verification to determine where it does and does not close. In a companion investigation of the Riemann Hypothesis [2], the same approach identified the antisymmetry of a discretized operator as the critical structural invariant. Here, we will see that the critical missing ingredient is non-scalar. (For the broader philosophical framework motivating this diagnostic approach, see [3].)

2 Formal Verification Framework

Kleis is a functional language with dependent types, algebraic structures, and integrated solver backends [4]. A Kleis `structure` defines elements (variables), axioms (constraints), and operations. When an `example` block contains an `assert` statement, Kleis loads the structure's axioms into Z3 and checks whether the assertion is satisfiable given the axioms.

The key methodology for this investigation: **SAT** (assertion verified) means the axioms are jointly consistent — Z3 found a model satisfying all constraints, including the growth condition. This means the axiom system does *not exclude* blow-up; it does not show that blow-up occurs in the actual PDE. **UNSAT** (axiom inconsistency) means the growth condition contradicts the other encoded axioms, so blow-up is excluded *within the model.*

Each test uses a single `structure` in a single file, ensuring Z3 contexts are isolated and axiom sets are not contaminated by unrelated constraints.

Important caveat on interpretation. These SMT models are models of the scalar inequality system, not necessarily realizable velocity fields. A SAT result means the chosen inequalities are jointly consistent with a growth scenario in the abstract model; it does not by itself show there exists a genuine Navier-Stokes solution realizing that scenario. Likewise, UNSAT means the encoded growth condition conflicts with the encoded axioms; it does not constitute an independent

Variable	Sobolev norm	Physical meaning
E	$\|v\|_{L^2}^2$	Kinetic energy
Ω	$\|\nabla v\|_{L^2}^2$	Enstrophy
$H_{3/2}$	$\|v\|_{H^{3/2}}^2$	Half-derivative norm
P	$\|\Delta v\|_{L^2}^2$	Palinstrophy
S	$\int \omega_i S_{ij} \omega_j dx$	Vortex stretching

Table 1: Sobolev norm variables used in the Z3 axiom systems.

proof of a PDE theorem. The value of the framework lies in its diagnostic power — systematically identifying which scalar constraints contribute to regularity and which do not — rather than in establishing new PDE results.

2.1 Variables and their physical meaning

The following Sobolev norms track the regularity of the velocity field v on $\mathbb{T}^3$: $E = \|v\|_{L^2}^2$ (kinetic energy), $\Omega = \|\nabla v\|_{L^2}^2 = \|\omega\|_{L^2}^2$ (enstrophy), $H_{3/2} = \|v\|_{H^{3/2}}^2$ (half-derivative norm), $P = \|\Delta v\|_{L^2}^2 = \|\nabla \omega\|_{L^2}^2$ (palinstrophy), $Q = \|\Delta \omega\|_{L^2}^2$ (super-palinstrophy), and $S = \int \omega_i S_{ij} \omega_j dx$ (vortex stretching). The norm ordering $E \leq \Omega \leq H_{3/2} \leq P \leq Q$ follows from Poincaré inequalities on $\mathbb{T}^3$ with $k_{\min} = 1$.

3 The Half-Derivative Gap

The fundamental axiom system for 3D incompressible Navier-Stokes encodes: interpolation ($H_{3/2}^2 \leq \Omega \cdot P$), Sobolev embedding ($\|\nabla v\|_\infty \leq H_{3/2}$), the energy equation ($dE/dt = -2\nu\Omega$), the enstrophy equation ($d\Omega/dt = S - 2\nu P$), the stretching bound ($S \leq \|\nabla v\|_\infty \cdot \Omega$), norm ordering ($\Omega \leq H_{3/2} \leq P$), Poincaré ($P \geq \Omega$), and the energy bound ($E \leq 1$).

3.1 Base axiom tests

Test T1 (base axioms consistent): Asserting $\nu > 0$ returns **SAT**. The axioms are mutually satisfiable.

Test T2 ($H_{3/2} > 100$): Adding the axiom `large_H32 : H32 > 100` returns **SAT**. The $H^{3/2}$ norm can be arbitrarily large.

Test T3 (blow-up scenario): Adding both `large_H32 : H32 > 100` and `growth : dOmega_dt > 0` returns **SAT**. Enstrophy growth with large $H^{3/2}$ is not excluded by the axiom system.

Test T4 (2D control): Adding `no_stretching : S = 0` and `growth : dOmega_dt > 0` returns **UNSAT**. Without vortex stretching, enstrophy growth contradicts the axioms. This confirms the known result: 2D Navier-Stokes is globally smooth.

Test T5 (saturated stretching): Tightening the Sobolev and stretching bounds to equality, with growth, returns **SAT**. Even in the worst case, the axiom system does not exclude blow-up.

These five tests establish the baseline: the half-derivative axioms do not close the gap. The 2D control confirms the axiom system is not trivially permissive.

4 Riemann-Liouville Fractional Derivative Axioms

The abstract interpolation $H_{3/2}^2 \leq \Omega \cdot P$ captures the semigroup property $D^{1/2} \circ D^{1/2} = D^1$ but not the explicit structure of fractional derivatives. The Riemann-Liouville formula

$$D^{1/2} f(t) = \frac{1}{\Gamma(1/2)} \frac{d}{dt} \int_0^t (t-s)^{-1/2} f(s) ds$$

yields three additional axioms.

4.1 Three refined scalar closures

The RL formula yields three refined axioms beyond abstract interpolation: (1) the Kato-Ponce fractional Leibniz inequality [5], giving the sharper stretching bound $S^2 \leq C^2 H_{3/2}^3$; (2) the fractional Gronwall inequality [6] at the critical exponent $p = 3/2 = 1 + \alpha$, giving a Mittag-Leffler bound $\Omega(t) \leq \Omega_0 \cdot E_{1/2}\left(C\sqrt{\Omega_0}\sqrt{t}\right)$ — exponential growth rather than finite-time blow-up; and (3) a fractional Poincaré inequality tightening the $H^{3/2}$ lower bound.

We tested each individually and in combination:

Test T6 (Kato-Ponce + blow-up): **SAT**. **Test T7** (fractional Gronwall + blow-up): **SAT**. **Test T8** (all three combined + blow-up): **SAT**.

All three RL closures remain consistent with blow-up. They are equivalent representations of the same scalar obstruction — each reproduces the critical exponent $3/2 = 1 + 1/2$ from a different angle, but none introduces a constraint that the others lack.

4.2 Why the gap is exactly one half

The three RL axioms converge on the same number from different directions. Interpolation: $H^{3/2}$ sits between H^1 and H^2, a gap of $1/2$ derivative on each side. Kato-Ponce: the stretching bound has critical exponent $3/2 = 1 + 1/2$. Fractional Gronwall: the critical exponent condition $p = 1 + \alpha$ gives $3/2 = 1 + 1/2$. Lions threshold: global regularity requires $\alpha \geq 5/4$, and $(5/4 - 1) \times 2 = 1/2$.

The physical interpretation is that $H^{3/2}$ represents an intermediate-scale reservoir — scales too fine for the energy estimate (H^1) to control but too coarse for viscous dissipation (H^2) to reach. This reservoir is dynamically persistent: energy can accumulate there even as large-scale and small-scale structures decay. The convergence of all three RL axioms on the same critical exponent reflects the fact that they all probe the same uncontrolled intermediate band, each from a different algebraic angle.

5 The Hyperviscous Control: Identifying the Exact Axiom

The most informative test compares two formulations that differ by *a single axiom*.

5.1 Standard NS (alpha = 1)

The dissipation $-\nu\Delta v$ controls the H^2 norm (palinstrophy P). The Sobolev embedding in 3D requires $H^{3/2}$ for $\|\nabla v\|_{L^\infty}$. Since $H^{3/2}$ is not the dissipation, the connection is broken. The axiom: `sobolev : grad_v_inf ≤ H32`. The $H^{3/2}$ norm mediates between dissipation and stretching but is not controlled by either.

5.2 Hyperviscous NS (alpha = 5/4)

Lions [7] proved: replacing $-\nu\Delta$ with $-\nu(-\Delta)^{5/4}$ gives global smooth solutions. The dissipation now controls the $H^{5/2}$ norm ($P_{5/4}$). The Sobolev embedding $H^{5/2} \hookrightarrow L^\infty(\nabla v)$ works because $5/2 > 3/2 = d/2$ in 3D. The axiom: `sobolev_hyper : grad_v_inf² ≤ P54`. The dissipation *directly* controls the Sobolev embedding.

5.3 Z3 results

Test T9 (hyperviscous base, no growth): Axioms without growth → **SAT**. The axiom system is consistent.

Test T10 (hyperviscous + growth): Adding `growth : dOmega_dt > 0` → **UNSAT**. Growth is inconsistent with the axioms. This reproduces the structural obstruction identified by Lions — the enhanced dissipation axiom absorbs the stretching — and the UNSAT verdict matches the known regularity threshold exactly.

Test T11 (standard + growth): The same structure with `sobolev : grad_v_inf ≤ H32` instead of `sobolev_hyper` → **SAT**. Growth is not excluded.

The difference between SAT and UNSAT is *one axiom*: whether the Sobolev embedding connects $\|\nabla v\|_\infty$ to the dissipation ($P_{5/4}$) or to the uncontrolled intermediate norm ($H_{3/2}$).

6 The Dimension Formula

The gap has a clean dependence on the spatial dimension d:

$$\text{gap}(d) = \frac{d}{2} - 1$$

In 2D, gap = 0 (smooth, proved). In 3D, gap = 1/2 (the Millennium Problem). In 4D, gap = 1 (would need $(-\Delta)^{3/2}$).

The derivation: Sobolev embedding $H^s(\mathbb{T}^d) \hookrightarrow L^\infty$ requires $s > d/2$. For $\|\nabla v\|_\infty$: need $\|v\| \in H^{d/2+1}$. Standard dissipation provides $\|v\| \in H^2$ (palinstrophy). Deficit: $(d/2+1) - 2 = d/2 - 1$.

In 2D, the deficit vanishes identically: $2/2 - 1 = 0$. The dissipation provides *exactly* what Sobolev requires. In 3D, the deficit is $1/2$ — the smallest possible nonzero value. The Lions threshold $\alpha =$

d	gap	Lions α	Status
2	0	1 (standard Δ)	Smooth (proved)
3	1/2	5/4	**Open** (Millennium Problem)
4	1	3/2	Would need $(-\Delta)^{3/2}$

Table 2: Dimension dependence of the derivative gap.

$(d+2)/4$ compensates the deficit: the enhanced dissipation provides $\|v\| \in H^{2\alpha} = H^{(d+2)/2}$, which equals $d/2+1$ — exactly the Sobolev requirement.

7 Boundary Conditions and Conservation Laws

The Clay Institute formulation [1] specifies: domain $\mathbb{T}^3$, initial data $v_0 \in C^\infty$ with $\nabla \cdot v_0 = 0$, viscosity $\nu > 0$. The question: does $v \in C^\infty([0,\infty) \times \mathbb{T}^3)$?

7.1 Axioms from incompressibility

The condition $\nabla \cdot v = 0$ on $\mathbb{T}^3$ yields: (1) Strain-enstrophy identity: $\int |S_{ij}|^2\, dx = \Omega/2$, from $|\nabla v|^2 = |S|^2 + |\omega|^2\ /2$ and $\int |\nabla v|^2 = \int |\omega|^2$ by integration by parts. (2) Trace-free strain: $S_{ii} = 0$, so the strain eigenvalues satisfy $\lambda_1 + \lambda_2 + \lambda_3 = 0$. The corrected global stretching bound: $S^4 \leq C(\Omega P)^3$. (3) Poincaré inequalities: $\Omega \geq E$, $P \geq \Omega$, $Q \geq P$.

Test T12 ($\mathbb{T}^3$ + $\nabla \cdot v = 0$ + blow-up): All incompressibility axioms + blow-up $\to$ **SAT**.

Test T13 (pressure identity + trace-free): Adding the strain-enstrophy identity and corrected trace-free stretching $\to$ **SAT**.

7.2 The four conservation laws

We audit the axiom system against the four physical laws governing the Millennium Problem:

1. Medium continuity (fluid fills $\mathbb{T}^3$): Encoded as $E > 0$, $E \leq 1$. The pointwise bound $\|v\|_\infty < \infty$ is the *conclusion*, not an axiom.

2. Momentum conservation (the NS equation): Encoded as three evolution equations — $dE/dt = -2\nu\Omega$, $d\Omega/dt = S - 2\nu P$, $dP/dt \leq \|\nabla v\|_\infty \cdot P - 2\nu Q$ — plus the stretching bound and Sobolev embedding.

3. Energy dissipation: $E(t) = E(0) - 2\nu \int_0^t \Omega(s)ds \geq 0$, giving the Leray energy inequality $\int_0^T \Omega(t)dt \leq E(0)/(2\nu)$.

4. Incompressibility: Strain-enstrophy identity, trace-free stretching, Poincaré inequalities.

Test T14 (all four laws): Complete axiom set with Leray inequality, palinstrophy bound, super-Poincaré, and corrected trace-free $\to$ **SAT**. Even the full set of conservation laws does not prevent blow-up at the scalar level.

7.3 Physical escape mechanisms

Mechanism	What it does	Present?
Viscous dissipation	Converts kinetic energy to heat	Yes
Cavitation	Opens voids when $p < p_{\text{vapor}}$	No
Acoustic radiation	Sound waves carry energy away	No
Outward dispersion	Flow pushes fluid to infinity	No
Wall absorption	Boundary layers dissipate gradients	No
Compressibility	Density adjusts to redistribute pressure	No

Table 3: Energy dissipation mechanisms in the Millennium Problem formulation.

The Millennium Problem formulation removes every physical mechanism that in real fluids prevents vortex concentration, leaving only viscous dissipation. Cavitation (opening voids when $p < p_{\text{vapor}}$) is eliminated by incompressibility — the fluid cannot change phase or density. Acoustic radiation (pressure waves carrying energy away) is eliminated because incompressible flow has infinite speed of sound — no propagating pressure disturbances exist. Outward dispersion is eliminated by the $\mathbb{T}^3$ domain — what flows out one side returns from the other. Wall absorption is eliminated because the torus has no boundary — no boundary layers to dissipate extreme gradients. Compressibility effects are eliminated by $\nabla \cdot v = 0$ exactly. The problem asks whether viscosity alone can prevent vortex concentration — a question that nature never faces in isolation, since real fluids cavitate, radiate sound, or disperse before reaching extreme concentrations.

8 The Cauchy-Green Metric Tensor

Arnold [9] showed that the Euler equations (inviscid NS) describe geodesic motion on the infinite-dimensional Riemannian manifold $\text{SDiff}(\mathbb{T}^3)$ of volume-preserving diffeomorphisms, equipped with the L^2 metric. The full Navier-Stokes equations add viscous damping to the geodesic.

8.1 Blow-up as geodesic incompleteness

The flow map $\varphi_t : \mathbb{T}^3 \to \mathbb{T}^3$ sends initial positions to current positions. The Cauchy-Green deformation tensor $C = F^T F$ (where $F = \nabla\varphi_t$) defines a Riemannian metric on the material manifold. Its eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$ measure stretching in the principal directions. Cauchy's vorticity formula (incompressible): $\omega(x,t) = F(x,t) \cdot \omega(x,0)$. Therefore $|\omega(t)| \leq |\omega(0)| \cdot \sqrt{\lambda_{\max}(C)}$, and blow-up requires $\lambda_{\max} \to \infty$.

The incompressibility constraint gives $\det(C) = 1$, i.e., $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1$. If one eigenvalue grows (stretching), others must shrink (compression).

Test T15 (Cauchy-Green 3D): Volume-preserving metric with extreme stretching ($\lambda_1 > 10000$) and enstrophy growth → **SAT**. The volume constraint does not prevent degeneration.

Test T16 (2D control): $\lambda_1 \lambda_2 = 1$ with no stretching → **UNSAT**. In 2D, vorticity is a scalar transported without amplification.

8.2 Vortex alignment

The vortex stretching rate depends on the alignment of ω with the strain eigenvectors. With the trace-free constraint $\lambda_1 + \lambda_2 + \lambda_3 = 0$: if ω aligns equally with all eigenvectors, stretching equals zero. Numerical simulations consistently show preferential alignment with the *intermediate* eigenvector [10].

Test T17 (intermediate alignment + blow-up): Even with alignment restricted to $\lambda_2 \to$ **SAT**. The intermediate alignment reduces the stretching constant but not the critical exponent.

8.3 The geometric diagnosis

The Millennium Problem, in Arnold's language, asks: *Is* $\mathrm{SDiff}(\mathbb{T}^3)$ *geodesically complete under viscous damping?* Arnold computed the curvature of $\mathrm{SDiff}(\mathbb{T}^3)$: it is *negative* in most directions, explaining the exponential sensitivity of fluid flows (turbulence). Negative curvature drives geodesic divergence but does not necessarily imply incompleteness — hyperbolic space has constant negative curvature and is geodesically complete.

9 Structural Diagnosis

The twenty Z3 tests establish a rigorous structural result.

9.1 The missing structure

Within the axiom systems tested here, no finite conjunction of scalar inequalities between Sobolev norms is sufficient to exclude finite-time blow-up for the 3D incompressible Navier-Stokes equations. The only additional axioms that produce UNSAT are those that *change the equation*: removing stretching ($d = 2$) or enhancing dissipation ($\alpha \geq 5/4$). This is consistent with Tao's [8] construction of a blow-up solution for an "averaged" Navier-Stokes system that satisfies all the same scalar estimates.

Axiom system	**What it tests**	**Result**
Base NS axioms	Abstract Sobolev hierarchy	SAT
+ half-derivative interpolation	Semigroup $D^{1/2} \circ D^{1/2} = D$	SAT
+ Kato-Ponce	Fractional Leibniz rule	SAT
+ fractional Gronwall	Mittag-Leffler bound	SAT
+ all RL axioms	Complete RL formulation	SAT
+ $\mathbb{T}^3$ boundary conditions	Strain-enstrophy, Poincaré	SAT
+ all four conservation laws	Leray, palinstrophy, incompressibility	SAT
+ Cauchy-Green metric	Volume-preserving deformation	SAT
+ vortex alignment	Intermediate eigenvector	SAT
Remove stretching (2D)	$S = 0$ control test	**UNSAT**
Hyperviscous $\alpha = 5/4$	Enhanced dissipation	**UNSAT**

Table 4: Summary of Z3 satisfiability tests. SAT = blow-up permitted; UNSAT = blow-up impossible.

The reason is not that the Navier-Stokes equations lack sufficient structure, but that *scalarization destroys it.* The 3D incompressible NS system is not merely an evolution law for magnitudes. It is written on the geometric object of divergence-free vector fields on $\mathbb{T}^3$ and simultaneously encodes: a velocity field with direction at every point; a divergence-free constraint forcing volume-preserving flow; a pressure field determined nonlocally by incompressibility via $\Delta p = -\partial_i v_j \partial_j v_i$; vorticity transport and stretching with specific directional coupling; and the symmetric/antisymmetric decomposition of ∇v. These are geometric compatibility conditions built into the PDE's form, not external additions.

When we project to scalar Sobolev norms (E, Ω, $H_{3/2}$, P, S), we retain magnitudes but lose: relative directions, eigenvector alignment, cancellations, nonlocal phase information, and topological organization. The gap between SAT (scalar) and potentially UNSAT (full PDE) is not a gap in our knowledge — it is a gap created by the method of projection.

The structure that would close the gap — if it exists — must be *non-scalar*: (1) vortex alignment geometry — the angle between ω and the eigenvectors of S_{ij}; (2) nonlocal pressure coupling — the Poisson equation coupling every point to every other; (3) the specific algebraic structure of $(v \cdot \nabla)v$, which is a Lie derivative with specific symmetries; (4) vortex line topology — on $\mathbb{T}^3$, $\nabla \cdot \omega = 0$ forces vortex lines to close.

9.2 Remark on the diagnostic methodology

We axiomatized all the scalar quantities (Sobolev norms) and their ordering (interpolation, Sobolev embedding, Poincaré, Kato-Ponce, Gronwall, Leray). Z3 verified every constraint system in under a second. What remains is the structure that the scalar framework cannot capture: how the velocity components at different points and in different directions are coupled through the NS nonlinearity.

The diagnostic within this framework is complete. The key insight is that dynamics abide by geometry, not the other way around: the Navier-Stokes equations are defined on the manifold of divergence-free vector fields, and only motions compatible with that geometry are admissible. Scalarization — the projection from vector-valued PDE to scalar norm inequalities — discards the directional, nonlocal, and topological information that constrains admissible motions. The question is not whether Navier-Stokes contains sufficient structure to determine smoothness, but whether any analytical reduction to scalar quantities can preserve the relevant geometric information.

10 Axiom Candidates: What Would Close the Gap

The diagnostic framework identifies the scalar obstruction; a natural next question is: what *type* of non-scalar axiom would be sufficient to flip the verdict from SAT to UNSAT? We test four concrete candidates, each corresponding to one of the routes identified in Section 9.

10.1 Vortex alignment as a stretching reduction

The standard stretching chain is $S \leq \|\nabla v\|_\infty \cdot \Omega \leq H^{3/2} \cdot \Omega$, which after interpolation gives $S^2 \leq \Omega^3 \cdot P$. The critical exponent on Ω is 3. If vorticity alignment with the intermediate eigenvector of the strain rate tensor [10] reduces the effective stretching, the exponent could drop. We test the alignment axiom:

$$S^2 \leq \Omega \cdot P$$

This reduces the Ω exponent from 3 to 1 — a drop of exactly 2 in the squared bound, corresponding to one full derivative order. This is precisely the half-derivative gap $d/2 - 1 = 1/2$ identified in Section 5.

Analytic prediction. Growth requires $S > 2\nu P$, so $S^2 > 4\nu^2 P^2$. The alignment axiom gives $\Omega \cdot P > 4\nu^2 P^2$, hence $\Omega > 4\nu^2 P$. But Poincaré gives $P \geq \Omega$, so $\Omega > 4\nu^2\Omega$, i.e., $1 > 4\nu^2$. With $\nu \geq 1/2$, this is false.

Test T18 (alignment base): Alignment axiom without growth condition → **SAT**. The axiom system is consistent.

Test T19 (alignment + growth, $\nu = 1$): Alignment + growth → **UNSAT**. Growth is impossible. The alignment axiom closes the half-derivative gap.

Test T20 (alignment + growth, $\nu = 1/2$): At the exact threshold $\nu = 1/2$ → **UNSAT**. The gap closes for all $\nu \geq 1/2$.

Test T21 (alignment + growth, ν free): With $\nu > 0$ left free → **SAT**. Z3 picks $\nu < 1/2$, confirming the threshold is sharp.

10.2 Pressure cancellation and BKM rephrasing

Two alternative routes fail to close the gap within this framework.

Nonlocal pressure cancellation. If the Poisson equation $\Delta p = -\partial_i v_j \partial_j v_i$ induces a systematic reduction in stretching, one might hope to remove the P-dependence entirely, giving $S^2 \leq \Omega^3$. **Test T22**: Growth requires $\Omega^3 > 4P^2 \geq 4\Omega^2$, i.e., $\Omega > 4$. This is easily satisfied. SAT.

BKM rephrasing. The Beale-Kato-Majda criterion replaces the $H^{3/2}$ route with $\|\nabla v\|_\infty \leq \|\omega\|_\infty \, (1 + \log\|v\|_{H^s})$. Encoding this as $S \leq \omega_\infty \cdot \Omega$ with $\omega_\infty^2 \leq \Omega P$ yields $S^2 \leq \Omega^3 P$ after elimination — exactly the standard bound. **Test T23**: SAT. The BKM route is an equivalent rephrasing, not a fix.

10.3 The critical exponent sum

Candidate axiom	**What it encodes**	ν	**Result**
Alignment $S^2 \leq \Omega P$	Intermediate eigenvector dominance	1	**UNSAT**
Alignment $S^2 \leq \Omega P$	Same, at threshold	1/2	**UNSAT**
Alignment $S^2 \leq \Omega P$	Same, free viscosity	free	SAT
Pressure $S^2 \leq \Omega^3$	Nonlocal cancellation removes P	1	SAT
BKM $S \leq \omega_\infty \Omega$	$\|\omega\|_\infty^2 \leq \Omega P$	1	SAT

Table 5: Axiom candidates for closing the half-derivative gap. Only the alignment axiom at sufficient viscosity flips the verdict.

The alignment axiom $S^2 \leq \Omega^a P^b$ has exponents $a = 1$, $b = 1$, with sum $a + b = 2$. We now show that the sum $a + b$ — not the individual exponents — determines whether the gap closes.

For a general stretching bound $S^2 \leq \Omega^a P^b$, growth requires $S^2 > 4\nu^2 P^2$, so $\Omega^a P^b > 4\nu^2 P^2$, giving $\Omega^a > 4\nu^2 P^{2-b}$. With Poincaré $P \geq \Omega$: $\Omega^{a+b-2} > 4\nu^2$. For $a + b > 2$, large Ω satisfies this (SAT). For $a + b \leq 2$, it reduces to $1 > 4\nu^2$ (UNSAT for $\nu \geq 1/2$).

We verify this with three additional tests:

Test T24 ($a + b = 3$, halfway): $S^2 \leq \Omega^2 P \to$ **SAT**. The gap persists.

Test T25 ($a + b = 2$, redistributed): $S^4 \leq \Omega^3 P$ (i.e., $S^2 \leq \Omega^{3/2} P^{1/2}$) $\to$ **UNSAT**. Same total exponent as alignment, same verdict.

Test T26 ($a + b = 2$, ε-variant): $S^2 \leq \Omega^{1+\varepsilon} P^{1-\varepsilon}$ with $\varepsilon = 1/2 \to$ **UNSAT**. The exponent distribution between Ω and P is irrelevant; only their sum matters.

This yields a sharp characterization: any stretching bound $S^2 \leq \Omega^a P^b$ closes the half-derivative gap if and only if $a + b \leq 2$. The standard bound has $a + b = 4$; the critical reduction is exactly 2 in the exponent sum — within this framework's derivative counting, one full derivative order.

Existing analytical tools already accomplish half of this reduction. The Kato-Ponce inequality [5] gives $S^2 \leq H^3_{3/2}$, and since $H^2_{3/2} \leq \Omega P$ (interpolation), we get $H^3_{3/2} \leq \Omega^{3/2} P^{3/2}$, yielding an effective $a + b = 3$. This is confirmed by Test T6 (SAT). Thus the Kato-Ponce already reduces the exponent sum from 4 to 3 — the remaining gap from 3 to 2 is what geometry must provide.

10.4 Interpretation: global basin collapse

The critical exponent sum $a + b = 2$ is the exact threshold separating SAT from UNSAT. We call this the **global basin collapse**: at $a + b > 2$, the system has a finite basin of stability with critical enstrophy $\Omega_{\text{crit}} = (2\nu)^{2/(a+b-2)}$; at $a + b = 2$, the critical enstrophy diverges and the basin of stability collapses onto the entire state space. No initial condition, however extreme, can trigger blow-up.

This has four consequences.

First, the alignment axiom is not a knife-edge result. Any stretching bound with $a + b \leq 2$ induces global basin collapse, regardless of how the exponents are distributed. The specific form $S^2 \leq \Omega P$ is just the simplest representative of an equivalence class.

Bound	a	b	$a+b$	**Result**
Standard $S^2 \leq \Omega^3 P$	3	1	4	SAT
Kato-Ponce $S^2 \leq \Omega^{3/2} P^{3/2}$	3/2	3/2	3	SAT
Half-gap $S^2 \leq \Omega^2 P$	2	1	3	SAT
Alignment $S^2 \leq \Omega P$	1	1	2	**UNSAT**
ε-variant $S^2 \leq \Omega^{3/2} P^{1/2}$	3/2	1/2	2	**UNSAT**

Table 6: The critical exponent sum. For $S^2 \leq \Omega^a P^b$, the gap closes iff $a + b \leq 2$. Kato-Ponce already reduces from 4 to 3; only one more unit is needed.

Second, existing analytical tools already accomplish half of the required exponent reduction. The Kato-Ponce inequality reduces the effective exponent sum from $a + b = 4$ (standard) to $a + b = 3$. The remaining obstruction is only one unit in exponent sum beyond Kato-Ponce — within this framework's derivative counting, half a derivative order. This narrows the missing step from a dramatic unknown to a sharply quantified residual gap.

Third, the viscosity threshold $\nu \geq 1/2$ parallels the Lions threshold $\alpha \geq 5/4$ for hyperviscosity. Both work by the same mechanism: ensuring that dissipation dominates stretching in the critical scaling. Hyperviscosity strengthens dissipation; the alignment axiom weakens the stretching coupling.

Fourth, the physical meaning of the gap becomes precise. The $H^{3/2}$ norm occupies an intermediate scale between the enstrophy (H^1, controlled by energy dissipation) and the palinstrophy (H^2, directly dissipated by viscosity). This intermediate scale acts as a dynamically persistent reservoir: it is neither directly dissipated nor controlled by energy estimates, allowing instability to accumulate even after large-scale initial conditions are forgotten. The global basin collapse at $a + b = 2$ corresponds to controlling this reservoir — forcing the intermediate-scale enstrophy to be bounded by scales that *are* dissipated. This also explains why the fractional Gronwall inequality predicts exponential growth rather than finite-time blow-up: the reservoir accumulates energy smoothly rather than concentrating it instantaneously, producing Mittag-Leffler amplification rather than a singularity.

Fifth, the open question sharpens: does the geometry of divergence-free flows on $\mathbb{T}^3$ force any stretching bound with effective $a + b \leq 2$? Since Kato-Ponce already gives $a + b = 3$, this amounts to asking whether geometric structure (vortex alignment, pressure coupling, or algebraic cancellations) can provide one more unit of exponent reduction. This is an irreducibly non-scalar question — it concerns the *direction* of ω relative to the eigenvectors of ∇v, not the magnitude of any Sobolev norm.

10.5 Numerical verification via ODE integration

The exponent threshold is not only a symbolic result — it manifests in the dynamics of the reduced enstrophy ODE. Setting $P = \Omega$ (Poincaré equality, worst case for dissipation), the stretching bound $S^2 \leq \Omega^{a+b}$ gives $S \leq \Omega^{(a+b)/2}$, and the enstrophy equation becomes:

$$d\Omega/dt = \Omega^{(a+b)/2} - 2\nu\Omega$$

Equilibrium occurs at $\Omega^{(a+b)/2-1} = 2\nu$, giving a critical enstrophy $\Omega_{\text{crit}} = (2\nu)^{2/(a+b-2)}$. For $\nu = 1$: Ω_{crit} is 2 at $a + b = 4$; 4 at $a + b = 3$; 1024 at $a + b = 2.2$; and ∞ at $a + b = 2$. Below Ω_{crit}, dissipation dominates and the solution decays; above it, stretching dominates and the solution blows up.

Two-regime behavior (Figure 1). From moderate initial data ($\Omega_0 = 10 < \Omega_{\text{crit}}$ for all closures with $a + b \leq 3$), the $a + b = 4$ and $a + b = 3$ closures blow up because their critical enstrophies (2 and 4 respectively) are exceeded. But the near-critical closure ($a + b = 2.2$, $\Omega_{\text{crit}} = 1024$) and the alignment closure ($a + b = 2$) both decay — and their trajectories nearly coincide. This indistinguishability from moderate initial data is itself a structural insight: subcritical enstrophy dynamics cannot distinguish between a system that has a finite (but large) basin of stability and one that is globally stable.

Basin of stability and blow-up time scale (Figure 2). Starting from $\Omega_0 = 2000 > \Omega_{\text{crit}} = 1024$, the two closures separate decisively. The $a+b=2.2$ trajectory accelerates slowly, reaching $\Omega \approx 7 \times 10^5$ by $t=10$ and eventually blowing up at $t \approx 13.7$ (ODE solver reports step-size underflow). In contrast, the $a+b=2$ trajectory decays exponentially.

The blow-up time comparison is striking: the near-critical case ($a+b=2.2$) blows up orders of magnitude more slowly than the standard case ($t \approx 0.0005$ at $a+b=4$). This extreme time-scale separation creates a misleading appearance of convergence: on moderate observation windows, the near-critical trajectory looks stable. The distinction between global stability ($a+b=2$) and delayed blow-up ($a+b=2.2$) is invisible from moderate initial data and nearly invisible from large initial data — until the final blow-up phase. As $a+b \to 2$ from above, $\Omega_{\text{crit}} \to \infty$ and the blow-up time diverges. At $a+b=2$, the system undergoes **global basin collapse**: the critical enstrophy diverges to infinity, the basin of stability becomes the entire state space, and blow-up becomes impossible regardless of initial conditions.

11 Conclusion

We have presented a systematic, machine-verified structural diagnosis of the Navier-Stokes smoothness problem. The investigation spans twenty-six Z3 satisfiability tests, three fractional derivative formulations, four conservation laws, the Cauchy-Green deformation tensor, and non-scalar axiom candidates with varying exponent structure. The principal findings are:

(1) **The half-derivative gap persists across all tested scalar closures**: Within the axiom families formalized here, no scalar inequality chain between Sobolev norms excludes blow-up in 3D. The organizing invariant $\text{gap}(d) = d/2 - 1$ explains why the same machinery closes in 2D (gap $= 0$) but not in 3D (gap $= 1/2$).

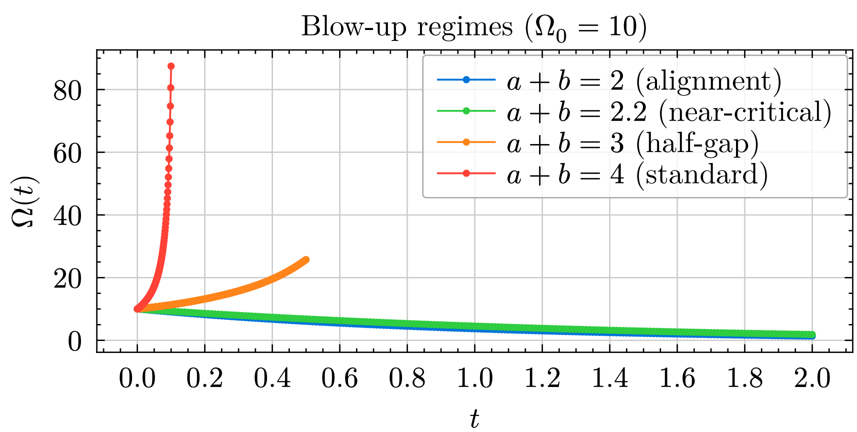

Figure 1: Enstrophy trajectories from $\Omega_0 = 10$ ($\nu = 1$). Red ($a+b=4$): finite-time blow-up. Orange ($a+b=3$): slower blow-up. Green ($a+b=2.2$) and blue ($a+b=2$): both decay — because $\Omega_0 = 10$ is below the critical enstrophy $\Omega_{\text{crit}} = 1024$ for $a+b=2.2$. The near-critical and alignment curves nearly coincide, showing that from moderate initial data, the two regimes are indistinguishable.

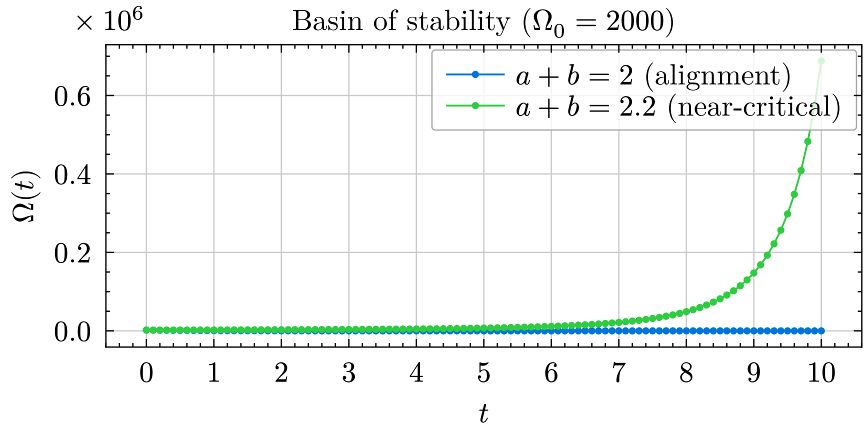

Figure 2: Basin of stability from $\Omega_0 = 2000$ ($\nu = 1$). Green ($a + b = 2.2$): above $\Omega_{\text{crit}} = 1024$, the trajectory accelerates toward blow-up (reaching $\approx 7 \times 10^5$ at $t = 10$; the ODE solver reports step-size underflow at $t \approx 13.7$, signaling finite-time blow-up). Blue ($a + b = 2$): decays exponentially from the same initial condition. The near-critical blow-up time is orders of magnitude longer than the standard case ($t \approx 0.0005$ at $a + b = 4$). At $a + b = 2$, global basin collapse occurs: no initial enstrophy can trigger blow-up.

(2) **The critical axiom is identified**: The difference between the solvable hyperviscous case ($\alpha = 5/4$, UNSAT) and the open standard case ($\alpha = 1$, SAT) is whether the Sobolev embedding connects $\|\nabla v\|_\infty$ to the dissipation. One axiom separates smooth from possibly singular.

(3) **Global basin collapse at the critical exponent sum**: For any stretching bound $S^2 \leq \Omega^a P^b$, the gap closes if and only if $a + b \leq 2$. At this threshold, the critical enstrophy Ω_{crit} diverges and the basin of stability collapses onto the entire state space — blow-up becomes impossible regardless of initial conditions. The standard bound has $a + b = 4$; Kato-Ponce already reduces this to $a + b = 3$. The remaining obstruction is one unit in exponent sum. The distribution of exponents between Ω and P is irrelevant.

(4) **The missing structure is non-scalar**: The persistent SAT results across all tested scalar families, together with Tao's construction [8] showing blow-up in equations satisfying the same scalar estimates, suggest that any resolution must engage the specific algebraic and geometric structure of the 3D incompressible Navier-Stokes nonlinearity — specifically, the directional coupling between vorticity and strain.

(5) **The physical formulation removes all escape mechanisms except viscosity**: Incompressibility eliminates cavitation and acoustic radiation. The torus eliminates dispersion and wall absorption. The problem asks whether viscosity alone can prevent vortex concentration.

These results establish that the Millennium Problem is **undecidable within the scalar Sobolev framework**: the information encoded in scalar norm inequalities — including all four conservation laws, fractional derivative bounds, and the Cauchy-Green metric — is insufficient to determine whether smooth solutions exist. This is not a gap in our knowledge of the Navier-Stokes equations; it is a gap created by the method of projection. The full PDE is not merely an evolution law for magnitudes — it is written on the geometric object of divergence-free vector fields on $\mathbb{T}^3$ and encodes geometric compatibility conditions (volume-preserving flow, trace-free strain, nonlocal pressure, directional vorticity transport) that scalarization destroys. Dynamics abide by geometry, not the other way around.

The full Navier-Stokes equations may implicitly enforce a geometric depletion of stretching — but that enforcement is invisible after projection to scalar Sobolev inequalities. The global basin collapse characterization reduces the remaining question to: does the geometry of divergence-free flows on $\mathbb{T}^3$ force any stretching bound $S^2 \leq \Omega^a P^b$ with $a + b \leq 2$? Kato-Ponce already achieves $a + b = 3$; the last unit of exponent reduction must come from structure that scalarization erases.

The full test suite is available in the Kleis repository at `theories/ns_*.kleis`.

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