The Mass Gap as a Classical Spectral Phenomenon: A Reinterpretation

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Abstract

We propose a reinterpretation of the Yang–Mills mass gap problem based on the conditional reduction established in Volume VIII of the Projected Ontology Theory (POT) program. The central observation is that the mass gap decomposes into two components of fundamentally different character: a *classical spectral mechanism*, expressed entirely through Sturm–Liouville theory, integral transform asymptotics, and inverse spectral analysis; and a *quantum realization problem*, requiring only a sign condition on an infrared anomalous dimension and the existence of a rigorous four-dimensional Yang–Mills theory. The classical component uses mathematics no later than 1946 (Watson, Weyl, Borg). No quantization, path integrals, gauge fixing, or particle physics appears in the mechanism. Quantum field theory contributes exactly one bit of information – the sign of a number – and one structural bridge. This separation reframes the mass gap from a purely quantum mystery into a problem of locating a classical mechanism within a quantum theory. The difficulty was never the mechanism. The difficulty is the realization.

Keywords: Yang-Mills mass gap, classical spectral theory, Sturm-Liouville, paradigm shift, mechanism vs realization, projected ontology

1 Introduction

The Yang–Mills mass gap is traditionally framed as a fundamentally quantum phenomenon: a non-perturbative feature of gauge theory requiring the full machinery of quantum field theory. The Clay Mathematics Institute formulation [1] demands a rigorous 4D Yang–Mills construction satisfying Wightman axioms with a positive mass gap. Decades of work have approached the problem through lattice simulations [2], functional renormalization [3], topological models [4], and AdS/CFT duality.

This essay advances a different perspective, based on the conditional reduction established in the companion paper [5] and the broader Projected Ontology Theory (POT) program [6, 7]:

$$\text{mass gap} = \underbrace{\text{classical spectral mechanism}}_{B+C+D} + \underbrace{\text{quantum realization}}_{A+E}.$$

The key point is not that quantum field theory is irrelevant. It is that its role is *narrower than previously assumed.* The mechanism that produces a spectral gap is classical. The challenge is whether Yang–Mills theory realizes that mechanism.

This separation is not a technicality. It is a reinterpretation of where the problem lives.

2 The Classical Spectral Mechanism

Consider a Sturm–Liouville operator on the half-line:

$$L = -\frac{d^2}{dx^2} + V(x), \quad x \in (0, \infty), \quad u(0) = 0.$$

If $V(x) \to +\infty$ as $x \to \infty$, the Rellich–Molchanov theorem [8] implies: the spectrum is purely discrete, eigenvalues accumulate only at $+\infty$, and the lowest eigenvalue $\lambda_1 > 0$. This is a *classical fact*. No quantum field theory is involved.

The reduction program [5] shows that a single scalar condition, $\gamma > 0$, propagates through a chain of classical results:

$$\gamma > 0 \Rightarrow \beta = 1 + \gamma > 1 \Rightarrow \alpha = \gamma > 0 \Rightarrow V(x) \sim x^{2\gamma} \to +\infty \Rightarrow \Delta > 0.$$

Each step uses only classical mathematics:

2.1 The ingredients

1. **Watson's lemma** (1918) [9]: transfers IR weight singularity to position-space kernel growth.
2. **Gauss hypergeometric equation** (1812) [10]: identifies the kernel as a Green's function of a Sturm–Liouville operator.
3. **Weyl semiclassical formula** (1911) [11]: connects potential growth to eigenvalue counting.
4. **Karamata's Tauberian theorem** (1930) [12]: inverts the spectral asymptotics.
5. **Abel inversion**: extracts the potential from the eigenvalue counting function.
6. **Borg uniqueness theorem** (1946) [13]: guarantees structural rigidity of the inverse spectral map.
7. **Rellich–Molchanov compactness** [8]: converts confining growth into discrete spectrum.
8. **Darboux transformations** (1882) [14]: generates the confining potential from a smooth superpotential.

The most recent result in this list is from 1946. The chain contains no Feynman diagrams, no path integrals, no gauge fixing, no second quantization, no operator algebras, no lattice discretization, no Monte Carlo simulation. The operator L is a second-order ODE on a half-line with a Dirichlet condition. In the language of 19th-century physics, it is a vibrating string with a growing restoring force.

3 Integral Transform Composition as Architecture

A central observation of the POT program, established in Volume VII [7], is that the relevant kernel is not merely computational – it is *structural*.

The integral transform composition method (ITCM) constructs a transmutation operator $T_w = H_\nu^{-1} \circ M_w \circ H_\mu$ from Hankel transforms H_μ, H_ν and a multiplication operator M_w. The resulting kernel

$$K(x, y) = \int_0^\infty w(k) J_\mu(kx) J_\nu(ky) k \, dk$$

has ${}_2F_1$ (Gauss hypergeometric) structure inherited from the Sonine–Poisson–Delsarte transmutation [15].

Volume VII [7] established three structural facts:

1. The kernel acts as a *resolvent*: $K(x, y; z) = (L - zI)^{-1}(x, y)$ for a Sturm–Liouville operator L determined by the transform composition (Theorem B, verified in `pot_theorem_b.kleis`).
2. The weight acts as a *spectral measure*: the IR behavior of $w(k)$ encodes the large-x asymptotics of the potential V.
3. The Hankel-order asymmetry $\mu \neq \nu$ is the *mass gap*: when $\mu = \nu$, the kernel reduces to the Bessel Green's function with continuous spectrum; when $\mu \neq \nu$, the weight singularity forces a confining potential and discrete spectrum.

In this view, integral transform composition is not a technique applied to a problem. It is the *underlying architecture* from which the operator, the potential, and the spectral gap all emerge. The Green's function is not imposed – it is *recognized*.

4 Where Quantum Field Theory Enters

Quantum field theory appears in exactly two places, and the reduction [5] is transparent about both.

4.1 Input A: The anomalous dimension

The gluon propagator in Yang–Mills theory has an infrared anomalous dimension γ. Three independent lines of evidence – perturbative QCD [16], lattice QCD [2], and Dyson–Schwinger equations [3] – support $\gamma > 0$.

The reduction uses *only the sign*. Not the value. Not the perturbative expansion. Not the renormalization group flow. One bit of information from quantum field theory: $\gamma > 0$.

This is the entire physics input to the classical mechanism.

4.2 Input E: The structural bridge

The existence of a rigorous 4D Yang–Mills theory – satisfying Wightman axioms [17] or the Osterwalder–Schrader reconstruction [18] – whose two-point function induces the ITCM kernel and Sturm–Liouville operator identified above.

This is the unresolved constructive QFT problem. It is the content of the Clay Millennium Prize.

Crucially, the *entire mechanism* connecting $\gamma > 0$ to $\Delta > 0$ is independent of E. The mechanism is complete without the bridge. What E provides is not the mechanism but the *realization*: the guarantee that Yang–Mills theory actually instantiates the classical spectral structure.

5 Mechanism versus Realization

This leads to a conceptual separation that we believe is the central contribution of the POT program:

Question	**Domain**
Why does a gap exist?	Classical spectral theory (B+C+D)
Whether Yang–Mills exhibits it	Quantum field theory (A+E)
How large is the gap?	Classical (Airy eigenvalues) + physics (γ value)

The mass gap is not intrinsically quantum in its *mechanism*. It is a classical spectral inevitability, contingent on a parameter supplied by QFT and a structural bridge that QFT must provide.

An analogy may clarify. Consider a violin string: a one-dimensional vibrating system with a fixed endpoint and a restoring force (tension). *Why does it produce discrete pitches?* The mechanism is classical – the eigenvalues of $-d^2/dx^2 + V(x)$ on an interval with Dirichlet boundary conditions are discrete. The *realization* depends on having a physical string under tension. Nobody would call the discreteness of the harmonic series a "material science phenomenon" simply because a physical string is needed to realize it. The mechanism is spectral theory. The realization is physics. Our operator is precisely this: a one-dimensional Sturm–Liouville equation on a half-line with a Dirichlet condition at the origin. The confining potential $V(x) \to \infty$ plays the role of tension. The mass gap is the fundamental frequency.

Similarly, the mass gap mechanism is spectral theory. The realization is quantum field theory. The historical difficulty of the mass gap problem arises, we suggest, from conflating the two.

6 Implications for the Mass Gap Problem

If this reinterpretation is correct, it has consequences for how the field allocates intellectual effort.

6.1 The problem is relocated, not solved

The Clay Millennium Problem asks for both the construction of a rigorous Yang–Mills theory *and* a proof that it has a mass gap. The reduction separates these: the mass gap follows from the construction (via A+E) plus the classical scaffold (B+C+D). The open problem is therefore:

Does a rigorous 4D Yang–Mills theory exist that realizes the classical spectral mechanism?

This is a different question from: *Why does a gap exist once such a mechanism is present?* The second question is answered by classical spectral theory. The first remains open.

6.2 Decades of work addressed the frame, not the mechanism

Much of the literature on confinement – dual superconductor models [19], center vortex models [20], Gribov copies [21] – attempts to explain *why* QCD confines. These are investigations of the *frame* (how Yang–Mills theory realizes confinement) rather than the *mechanism* (why confinement produces a gap once present).

The reduction suggests that the mechanism was always available in the classical literature. The spectral gap of a confining Sturm–Liouville operator is a theorem, not a conjecture. What was missing was not a calculation but a *viewpoint*: the recognition that the ITCM kernel carries Sturm–Liouville structure, and that Hankel-order asymmetry is the spectral gap in disguise.

6.3 No new physics was needed

Every mathematical ingredient in the reduction predates quantum field theory. Watson's lemma (1918), Weyl's asymptotic law (1911), the Gauss hypergeometric function (1812), Darboux transformations (1882), Borg's uniqueness theorem (1946), Sturm–Liouville theory itself (1836). The anomalous dimension $\gamma > 0$ is measured, not derived. The only genuinely quantum element is Assumption E – the existence problem – which is precisely the part left open.

The mass gap was never a quantum mystery. It was a classical spectral phenomenon that *appears* in a quantum theory.

7 Epistemological Shift

A secondary contribution is methodological. The reduction is built with explicit assumptions (A–E), labeled confidence levels (Level A through Level C/D), formal verification of logical steps (400+ Z3-checked examples across 14 Kleis theory files [22]), and explicit falsifiability conditions for each assumption.

This enforces a discipline where gaps are *identified*, not obscured; claims are conditional when they must be; and the reader knows exactly what would break the argument. The contribution is not only a chain of reasoning but a *transparent structure* of reasoning.

The traditional mode of mathematical physics – plausible narrative punctuated by "it can be shown that" – is replaced by machine-verified logic with assumption tracking. Whether or not the mass gap reduction survives scrutiny, this methodology stands on its own: a demonstration that formal verification can discipline speculative mathematical physics without sterilizing it.

8 Conclusion

The Yang–Mills mass gap problem can be reframed as:

$$\text{mass gap} = \underbrace{\text{classical spectral mechanism}}_{\text{Sturm--Liouville, Watson, Weyl, Borg}} + \underbrace{\text{quantum realization}}_{\text{anomalous dimension + 4D existence}}.$$

The classical component is fully analyzable with established mathematics. The remaining difficulty lies in constructing and validating the quantum field theory that realizes it.

This does not solve the Clay Millennium Problem. It *relocates* it. The mechanism that produces a spectral gap was never quantum. It was classical, available since the 1940s, distributed across spectral theory, asymptotic analysis, and inverse problems. What was missing was the architectural insight – provided by the ITCM framework and the POT program – that these classical pieces compose into a single pipeline from $\gamma > 0$ to $\Delta > 0$.

The mass gap was not hiding in quantum field theory. It was hiding in the spectral theory of second-order ODEs on a half-line. Quantum field theory is the *theater* in which the gap appears. Classical spectral theory is the *script*.

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