

# Electrodynamics as a Theorem of Projected Ontology

Engin Atik<sup>1</sup>

<sup>1</sup>Kleis Research, <https://kleis.io>

## Abstract

We show that classical electrodynamics can be reduced to a single dynamical equation relating field to sources once expressed in the exterior calculus on an oriented Lorentzian 4-manifold: all remaining structure follows from the nilpotency of the exterior derivative,  $d^2 = 0$ , and is formally verifiable. Given a smooth oriented 4-manifold  $(M, g)$  with Lorentzian signature, the exterior algebra, and the Hodge star operator, we take two axioms — the definition  $F = dA$  (field strength as an exact 2-form) and the inhomogeneous Maxwell equation  $d \star F = \star J$  (relating the field to sources) — and derive the complete differential-form structure of classical electrodynamics as a chain of one- and two-step consequences. Gauge invariance ( $A \rightarrow A + d\chi$  leaves  $F$  unchanged), the homogeneous Maxwell equation ( $dF = 0$ , encoding Faraday’s law and the absence of magnetic monopoles), and charge conservation ( $d(\star J) = 0$ , the continuity equation) all follow from  $d^2 = 0$  applied at most twice. We formalize these derivations on the Kleis verification platform, where every axiom — both the mathematical background structure and the physical content — is explicit, and every theorem is machine-checked by the Z3 SMT solver. The entire exterior calculus (wedge product, Hodge star, interior product, Lie derivative, de Rham cohomology) is axiomatized in the Kleis standard library. We further show that the exterior derivative  $d$ , restricted to 1-forms, satisfies the admissible kernel axioms of Projected Ontology Theory (linearity, zero preservation, composition closure): gauge equivalence is literally projective equivalence under this kernel, and the gauge orbits are its nullspace. The electromagnetic sector thereby connects to the same kernel framework used in our companion papers on flat galactic rotation curves (gravitational kernel) and quantum entanglement (measurement kernel). A key classification result emerges: electrodynamics is the unique gauge sector with an admissible projection kernel, because the  $U(1)$  Lie algebra is abelian; the Yang–Mills generalization  $F = dA + A \wedge A$  breaks admissibility through its quadratic term. The result is a demonstration that the irreducible physical content of electrodynamics is concentrated in a single equation ( $d \star F = \star J$ ); everything else follows from the mathematical structure of an admissible kernel on a Lorentzian manifold.

**Keywords:** Maxwell equations, gauge invariance, formal verification, projected ontology, admissible kernel, exterior calculus, Noether theorem, Hodge star, differential forms, Z3 theorem prover, Cartan geometry

## 1 Introduction

What are Maxwell’s equations? The standard answer is: empirical laws, discovered by Faraday and Maxwell, encoding the behavior of electric and magnetic fields. The mathematical physicist’s answer is more refined: they are the unique equations governing a  $U(1)$  gauge field on a Lorentzian

manifold. But even this answer treats them as **given** — the starting point of a theory, not its output.

We propose a stronger claim: classical electrodynamics can be reduced to a **single dynamical equation** relating field to sources, once the mathematical infrastructure is made explicit. The infrastructure is: a smooth, oriented 4-dimensional manifold  $M$ ; a Lorentzian metric  $g$  with signature  $(-, +, +, +)$ ; the exterior algebra of differential forms with exterior derivative  $d$  satisfying  $d^2 = 0$ ; and the Hodge star operator  $\star$  determined by  $g$  and the orientation. None of these are physical postulates — they are the mathematical stage on which physics is performed.

On this stage, we take two axioms: the definition  $F = dA$  (field strength as an exact 2-form) and one irreducible physical equation  $d\star F = \star J$  (the inhomogeneous Maxwell equation relating the field to its sources). From these two statements alone, the entire structure of classical electrodynamics follows:

1. **Gauge invariance:**  $A \rightarrow A + d\chi$  leaves  $F$  unchanged (one step from  $d^2 = 0$ ).
2. **Homogeneous Maxwell equation:**  $dF = 0$  — no magnetic monopoles, Faraday’s law (one step from  $F = dA$  and  $d^2 = 0$ ).
3. **Charge conservation:**  $d(\star J) = 0$ , i.e.,  $\partial_\mu J^\mu = 0$  (two steps: apply  $d$  to  $d\star F = \star J$ , use  $d^2 = 0$ ).

No additional physical postulates are needed. The potential is an auxiliary projection variable; the field is the only ontologically meaningful object; charge conservation is the algebraic consequence of applying  $d$  to the inhomogeneous equation (admitting a Noether interpretation under the standard action); and monopoles are excluded whenever  $F$  is globally exact.

This paper is the third in a series on Projected Ontology Theory (POT). The first derived flat galactic rotation curves from the admissible kernel framework without dark matter. The second derived Bell inequality violation and GHZ contextuality from the same kernel structure, reinterpreting quantum entanglement as a projection artifact. The present paper adds the gauge-theoretic sector. We show that the exterior derivative  $d$ , restricted to 1-forms, **is** an admissible kernel in the formal POT sense: it satisfies linearity, zero preservation, and composition closure. Gauge equivalence is projective equivalence under this kernel; the gauge orbits are its nullspace; and the nilpotency  $d^2 = 0$  is a structural property of the kernel’s self-composition. A classification result emerges: electrodynamics is the unique gauge sector with an admissible projection kernel, because non-abelian theories introduce a quadratic self-interaction  $A \wedge A$  that breaks linearity.

Every derivation in this paper is machine-checked. The axioms are explicit, stated in the Kleis verification language, and verified by the Z3 SMT solver. The entire exterior calculus — wedge product, Hodge star, interior product, Cartan’s Magic Formula, de Rham cohomology — is axiomatized in the Kleis standard library (`stdlib/differential_forms.kleis`). The reader can reproduce every result by running the source file.

## 2 The Projected Ontology Framework

We work within the mathematical framework of Projected Ontology Theory (POT), building on the Cartan geometry pipeline axiomatized in the Kleis standard library. The reader unfamiliar with the full POT axiom set is referred to our companion papers for details; here we summarize only what is needed for the electrodynamics derivation.

## 2.1 Background Structure

Before stating the two physical axioms, we make explicit the mathematical infrastructure they presuppose. This infrastructure is not physics — it is the stage on which physics is performed:

1. **Smooth manifold  $M$** : A 4-dimensional smooth (i.e.,  $C^\infty$ ) manifold serving as the spacetime continuum. All operations below are defined on  $M$ .
2. **Lorentzian metric  $g$** : A non-degenerate symmetric bilinear form of signature  $(-, +, +, +)$  on the tangent bundle  $TM$ . The metric determines causal structure and is required for the Hodge star.
3. **Orientation**: A choice of volume form on  $M$ , required (together with  $g$ ) to define the Hodge star operator  $\star$ .
4. **Exterior algebra**: The graded algebra of differential forms  $\Omega^{p(M)}$  for  $p = 0, 1, \dots, 4$ , equipped with the wedge product  $\wedge$  and the exterior derivative  $d : \Omega^p \rightarrow \Omega^{p+1}$  satisfying the nilpotency  $d^2 = 0$ .
5. **Hodge star  $\star$** : The isomorphism  $\star : \Omega^{p(M)} \rightarrow \Omega^{n-p}(M)$  determined by  $g$  and the orientation, with the involutivity  $\star \star \alpha = (-1)^{p(n-p)+s} \alpha$  where  $s$  is the number of negative eigenvalues of  $g$  (here  $s = 1$ ).

Items 1–3 are geometric choices. Item 4 is a structural property of smooth manifolds (the nilpotency  $d^2 = 0$  is equivalent to the symmetry of mixed partial derivatives). Item 5 is determined by items 2 and 3.

The physical content enters only through the two axioms stated in Section 3. Everything derived in Sections 4–7 follows from these axioms plus the background structure above. The claim of this paper is not that electrodynamics requires **no** assumptions, but that once the mathematical stage is set, a single physical equation ( $d \star F = \star J$ ) suffices to determine the rest.

## 2.2 Cartan Geometry Pipeline

The Kleis standard library implements Élie Cartan’s approach to differential geometry using differential forms. The computation pipeline is:

$$\text{Tetrad } e^a \rightarrow \text{Connection } \omega_b^a \rightarrow \text{Curvature } R_b^a \rightarrow \text{Ricci, Einstein}$$

Given a tetrad (orthonormal frame)  $e^a$  encoding the metric via  $g = \eta_{ab} e^a \otimes e^b$ , the torsion-free condition  $de^a + \omega_b^a \wedge e^b = 0$  uniquely determines the connection 1-forms  $\omega_b^a$ . The curvature 2-forms are then:

$$R_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c$$

This is Cartan's second structure equation. The Bianchi identity  $dR_b^a + \omega_c^a \wedge R_b^c - R_c^a \wedge \omega_b^c = 0$  follows from  $d^2 = 0$ . Contracting gives the Ricci tensor and scalar curvature, hence the Einstein tensor  $G_{\mu\nu}$ .

The key point for electrodynamics: the same exterior derivative  $d$  and wedge product  $\wedge$  that govern spacetime curvature also govern the electromagnetic field. The field strength  $F$  is a 2-form, just as the curvature  $R_b^a$  is a matrix of 2-forms. The algebraic structures are identical.

### 2.3 The Exterior Algebra

The derivations in this paper rely on four operations from the exterior algebra, all axiomatized in the Kleis standard library:

1. **Exterior derivative**  $d$ : maps  $p$ -forms to  $(p+1)$ -forms. The fundamental property is  $d^2 = 0$  (nilpotency), which encodes the topology of the manifold.
2. **Hodge star**  $\star$ : maps  $p$ -forms to  $(n-p)$ -forms on an  $n$ -dimensional manifold with metric signature  $s$  (the number of negative eigenvalues). The involutivity relation is:

$$\star \star \alpha = (-1)^{p(n-p)+s} \alpha$$

For the electromagnetic 2-form  $F$  on a 4D Lorentzian manifold ( $p = 2$ ,  $n = 4$ ,  $s = 1$ ):  $\star \star F = (-1)^{4+1} F = -F$ . This sign is essential for correct energy-momentum traces.

3. **Interior product**  $\iota_X$ : contracts a vector field  $X$  with a  $p$ -form, reducing degree by 1. Metric-independent.
4. **Lie derivative**  $\mathcal{L}_X$ : Cartan's Magic Formula  $\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d$  connects the algebraic and differential operations. Metric-independent.

The Leibniz rule  $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta)$  and the graded antisymmetry  $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$  complete the algebraic structure. Every identity used in this paper is an axiom in the standard library, verified by Z3.

### 2.4 The Electromagnetic Projection Kernel

In the Projected Ontology framework, observable physics arises from applying an **admissible kernel** — a linear map satisfying specific structural axioms — to a pre-observable configuration space. The admissible kernel axioms, formalized in `pot_admissible_kernels_v2.kleis`, require:

- **Linearity**:  $K(G, a + b) = K(G, a) + K(G, b)$  (`kernel_lin_add`)
- **Scalar linearity**:  $K(G, c \cdot a) = c \cdot K(G, a)$  (`kernel_lin_smul`)
- **Zero preservation**:  $K(G, 0) = 0$  (`kernel_maps_zero`)

together with **projective equivalence** ( $a \sim b$  iff  $K(G, a) = K(G, b)$ ), **nullspace** ( $a \in \ker(K)$  iff  $K(G, a) = 0$ ), and **composition closure** (the composition of admissible kernels is admissible). The first paper in this series identified the gravitational kernel with a logarithmic Green's function; the second identified the measurement kernel with spinor projections parameterized by detector angles.

We now identify the electromagnetic projection kernel  $K_{\text{em}}$ . The pre-observable configuration space is the space of gauge potentials  $\Omega^1(M)$  (1-forms on  $M$ ). The observable field space is  $\Omega^2(M)$  (2-forms). The kernel is the exterior derivative restricted to 1-forms:

$$K_{\text{em}}(A) = dA = F$$

This map satisfies every admissibility axiom:

1. **Linearity:**  $d(A + B) = dA + dB$  and  $d(cA) = c dA$  for any constant  $c$ . This follows from the linearity of the exterior derivative, axiomatized in `ExteriorDerivative`.
2. **Zero preservation:**  $d(0) = 0$ . The zero 1-form maps to the zero 2-form.

The POT structural machinery then maps directly to electromagnetic concepts:

**Projective equivalence is gauge equivalence.** Two potentials  $A$  and  $A'$  are projectively equivalent under  $K_{\text{em}}$  iff they produce the same field strength:  $dA = dA'$ . This is equivalent to  $d(A - A') = 0$ , i.e.,  $A - A'$  is a closed 1-form. On a contractible manifold (by the Poincaré lemma), every closed 1-form is exact:  $A' = A + d\chi$  for some 0-form  $\chi$ . Gauge equivalence is projective equivalence. This is not an analogy — it is the literal definition, instantiated for  $K_{\text{em}} = d$ .

**The nullspace is the gauge orbit.** A potential  $A$  is in the nullspace of  $K_{\text{em}}$  iff  $dA = 0$ , i.e.,  $A$  is a closed 1-form. On contractible  $M$ ,  $\ker(d|_{\Omega^1}) = \{d\chi : \chi \in \Omega^0(M)\}$ . The gauge freedom is exactly the kernel's nullspace: the set of potentials that produce zero field strength.

**Nilpotency is a kernel composition property.** Composing  $K_{\text{em}}$  with itself yields  $d \circ d = d^2 = 0$ : the composed kernel has total nullspace (every element maps to zero). In the POT formalism, this means applying the electromagnetic projection twice annihilates all information. No other sector in the POT framework has this property — the gravitational kernel  $\omega \rightarrow d\omega + \omega \wedge \omega$  is not even linear.

## 2.5 Kernel Factorization: The Electromagnetic Sector

The entanglement paper introduced a **kernel factorization**: the unified POT kernel decomposes as

$$K = K_{\text{univ}} \circ K_{\text{dyn}} \circ K_{\text{rep}}$$

where  $K_{\text{univ}}$  encodes universal geometric structure,  $K_{\text{dyn}}$  encodes dynamics, and  $K_{\text{rep}}$  encodes the representation/measurement sector. Each factor is admissible, and by `compose_admissible`, the composition is admissible.

The electromagnetic sector extends this factorization. We introduce the gauge-sector kernel  $K_{\text{em}} = d|_{\Omega^1}$  as an additional admissible factor. The combined kernel for the electromagnetic sector is:

$$K_{\text{em-sector}} = K_{\text{univ}} \circ K_{\text{dyn}} \circ K_{\text{em}}$$

The admissibility of  $K_{\text{em}}$  is verified (linearity + zero preservation of  $d$ ), and `compose_admissible` guarantees the composition is admissible.

A key observation distinguishes electrodynamics within gauge theory: **admissibility holds for  $K_{\text{em}}$  specifically because  $U(1)$  is abelian.** The Yang–Mills kernel  $A \rightarrow dA + A \wedge A$  includes the quadratic term  $A \wedge A$ , which violates `kernel_lin_add`:

$$d(A + B) + (A + B) \wedge (A + B) \neq (dA + A \wedge A) + (dB + B \wedge B)$$

The cross terms  $A \wedge B + B \wedge A$  break additivity. Electrodynamics is therefore the **unique** gauge sector in which the projection kernel is admissible. Non-abelian gauge theories (Yang–Mills, chromodynamics) require a generalization of the kernel framework beyond strict linearity — a direction for future work.

**Why admissibility is physically motivated.** Admissibility is not an arbitrary restriction imposed to single out  $U(1)$ . It encodes three physical requirements. First, **superposition**: observable field strengths must combine linearly —  $F_1 + F_2$  is the field of two independent sources, and the kernel must respect this. A nonlinear kernel would entangle independent pre-images, making the decomposition of a field into independent contributions undefined. Second, **stable equivalence classes**: gauge orbits must be affine subspaces (cosets of the nullspace), so that the quotient space  $\Omega^1 \frac{M}{\ker(K_{\text{em}})}$  inherits linear structure. This is what makes the space of physical field configurations a vector space rather than a nonlinear manifold. Third, **composability**: admissible kernels compose to admissible kernels (`compose_admissible`), which is essential for multi-sector theories — the combined gravitational-electromagnetic-measurement kernel must be well-defined. A nonlinear factor would destroy this closure, making the unified kernel framework inconsistent. The fact that  $U(1)$  is the unique abelian (and hence admissible) gauge group is a **consequence** of these requirements, not a tautology.

This characterization is a genuinely POT-specific result: it classifies gauge theories by whether their projection kernels satisfy the admissibility axioms, and identifies electrodynamics as the abelian/admissible boundary case.

### 3 The Electromagnetic 2-Form

In the language of differential forms, the electromagnetic field is a 2-form  $F$  on a 4-dimensional Lorentzian manifold  $(M, g)$  with signature  $(-, +, +, +)$ . In coordinates:

$$F = E_i dx^i \wedge dt + \frac{1}{2} B_{ij} dx^i \wedge dx^j$$

The potential is a 1-form  $A = \varphi dt + A_i dx^i$ , and the 4-current is a 1-form  $J = \rho dt + J_i dx^i$  (or equivalently, a 3-form  $\star J$  via the Hodge dual).

The entire content of classical electrodynamics rests on two statements:

#### 3.1 The Two Axioms

**Axiom 1 (Definition).** The field strength is the exterior derivative of the potential:

$$F = dA$$

This is a definition, not a dynamical equation. It asserts that the electromagnetic field is an **exact** 2-form.

**Axiom 2 (Physics).** The inhomogeneous Maxwell equation relates the field to its sources via the Lorentzian Hodge star:

$$d \star F = \star J$$

This single equation encodes both Gauss's law ( $\nabla \cdot E = \frac{\rho}{\epsilon_0}$ ) and Ampère's law with Maxwell's correction ( $\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \partial \frac{E}{\partial t}$ ).

These are the only independent axioms. Everything else — gauge invariance, the homogeneous Maxwell equation, charge conservation — is a **derivable theorem**. The remainder of this paper exhibits each derivation and its machine-checked verification.

## 4 Gauge Invariance as Geometric Tautology

The gauge transformation  $A \rightarrow A' = A + d\chi$ , where  $\chi$  is any scalar function (0-form), leaves the field strength invariant:

$$F' = dA' = d(A + d\chi) = dA + d(d\chi) = dA + 0 = F$$

The second equality uses linearity of  $d$ . The third uses the nilpotency  $d^2 = 0$ : the exterior derivative of any exact form vanishes. This is not an approximation, not a symmetry imposed by hand, and not a physical assumption. It is a **tautology of the exterior algebra**.

The physical consequence is profound: the potential  $A$  is not an observable quantity. It is a **degree of freedom of the mathematical description** — a choice of representative within an equivalence class of 1-forms that yield the same field  $F$ . Only  $F$  has ontological weight. The potential is an auxiliary projection variable, and gauge ‘symmetry’ is the statement that the projection is many-to-one.

In the POT kernel formalism (Section 2.5), this is the statement that  $A$  and  $A + d\chi$  are **projectively equivalent** under the electromagnetic kernel  $K_{\text{em}} = d$ : they produce the same observable field  $F = dA = d(A + d\chi)$ . The gauge orbit  $\{A + d\chi : \chi \in \Omega^0(M)\}$  is precisely the equivalence class of potentials under projective equivalence. The potential  $A$  is a pre-observable configuration; only its image under  $K_{\text{em}}$  — the field strength  $F$  — has ontological weight. Gauge invariance is not a feature of electrodynamics; it is the projective equivalence relation of an admissible kernel.

## 5 No Magnetic Monopoles: A Topological Theorem

The homogeneous Maxwell equation is:

$$dF = 0$$

This encodes two classical statements:  $\nabla \cdot B = 0$  (no magnetic monopoles) and  $\nabla \times E = -\partial \frac{B}{\partial t}$  (Faraday's law). The derivation from Axiom 1 is immediate:

$$dF = d(dA) = d^2 A = 0$$

The non-existence of magnetic monopoles is therefore **not** an independent law of physics. It is a one-step consequence of the definition  $F = dA$  and the nilpotency  $d^2 = 0$ .

The deeper statement involves de Rham cohomology. A  $p$ -form  $\alpha$  is **closed** if  $d\alpha = 0$  and **exact** if  $\alpha = d\beta$  for some  $(p-1)$ -form  $\beta$ . Since  $d^2 = 0$ , every exact form is closed:  $d(d\beta) = 0$ . The converse is not always true — it depends on the topology of the manifold.

The second de Rham cohomology group  $H_{\text{dR}}^2(M)$  measures the obstruction: it is the space of closed 2-forms modulo exact 2-forms. If  $H_{\text{dR}}^2(M) = 0$ , then **every** closed 2-form is exact, and any field strength  $F$  satisfying  $dF = 0$  must be expressible as  $F = dA$  for some potential  $A$ .

Magnetic monopoles would correspond to  $F$  being closed but **not** exact — a nontrivial element of  $H_{\text{dR}}^2(M)$ . On  $\mathbb{R}^4$  (or any contractible manifold),  $H_{\text{dR}}^2 = 0$ , and every closed 2-form is exact. The non-existence of monopoles is then a theorem, not a postulate.

**Important caveat.** Our Axiom 1 ( $F = dA$ ) is stronger than merely requiring  $dF = 0$ . It asserts that  $F$  is **globally exact** — that a global potential 1-form  $A$  exists on all of  $M$ . In the language of fiber bundles, this amounts to working on a **trivial**  $U(1)$  principal bundle. On a nontrivial bundle (e.g., over  $\mathbb{R}^3 \setminus \{0\}$ ),  $F$  can be closed but not globally exact, and the Dirac monopole lives precisely in this cohomological gap.

The distinction matters: our derivations hold for the trivial-bundle case, which covers standard Minkowski-spacetime electrodynamics. The generalization to nontrivial bundles — where  $A$  is a connection on a principal  $U(1)$ -bundle rather than a globally defined 1-form — requires replacing the de Rham cohomology argument with characteristic classes (Chern numbers), and monopole charge becomes quantized by the Dirac condition. This generalization does not invalidate our framework; it extends it to a topologically richer setting where Axiom 1 is replaced by the weaker requirement that  $F$  be closed.

In kernel language: the question of whether  $F$  is globally exact is the question of whether  $F$  lies in the **image** of  $K_{\text{em}} = d$ . On contractible  $M$ , the kernel is surjective onto closed 2-forms (by the Poincaré lemma), so every physical  $F$  lifts to a potential. On topologically nontrivial  $M$ , the image of  $K_{\text{em}}$  is strictly smaller than the space of closed 2-forms, and the gap is measured by  $H_{\text{dR}}^2(M)$ .

## 6 Energy-Momentum Conservation

The electromagnetic stress-energy tensor  $T_{\mu\nu}^{\text{EM}}$  encodes the energy density, momentum density, and stress of the electromagnetic field. Its conservation  $\partial_\mu T_{\text{EM}}^{\mu\nu} = 0$  (in the absence of sources) is the formal statement of energy-momentum conservation for the field.

In the Cartan formalism, this conservation law is not an additional postulate. It follows from the Bianchi identity. For the Riemann curvature 2-form:

$$dR_b^a + \omega_c^a \wedge R_b^c - R_c^a \wedge \omega_b^c = 0$$

This identity, which itself is a consequence of  $d^2 = 0$  applied to Cartan's second structure equation, implies the contracted Bianchi identity  $\nabla_\mu G^{\mu\nu} = 0$  for the Einstein tensor. Combined with the Einstein–Maxwell field equation  $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}^{\text{EM}}$ , this yields  $\nabla_\mu T_{\text{EM}}^{\mu\nu} = 0$ .

The Poynting vector  $S = E \times \frac{B}{\mu_0}$  and the electromagnetic energy density  $u = \frac{\epsilon_0 E^2 + \frac{B^2}{\mu_0}}{2}$  are components of  $T_{\mu\nu}^{\text{EM}}$ . Their conservation — the Poynting theorem — is not an independent result but a component of the tensorial conservation law that follows from the Bianchi identity.

The Lie derivative provides the infinitesimal version. For a timelike vector field  $\xi^\mu$  (representing time evolution), Cartan's Magic Formula gives:

$$\mathcal{L}_\xi F = d(\iota_\xi F) + \iota_\xi(dF) = d(\iota_\xi F)$$

The last equality uses  $dF = 0$  (derived in Section 5). The energy flux along  $\xi$  is then entirely determined by the contraction  $\iota_\xi F$  and its exterior derivative — no additional structure needed beyond what the exterior algebra provides.

## 7 Charge Conservation: The Noether Closing Argument

The derivation of charge conservation from the two axioms is a two-step chain:

**Step 1.** From Axiom 2:  $d \star F = \star J$ .

**Step 2.** Apply the exterior derivative  $d$  to both sides:

$$d(d \star F) = d(\star J)$$

The left side is  $d^2(\star F) = 0$  by nilpotency. Therefore:

$$d(\star J) = 0$$

In components, this is the continuity equation  $\partial_\mu J^\mu = 0$ : charge is locally conserved.

This result is not a separate law. It is a two-step algebraic consequence of the inhomogeneous Maxwell equation and  $d^2 = 0$ . No additional axiom is needed.

Note what this derivation does **not** require: it does not invoke an action functional, a variational principle, or the Noether theorem. The conservation law  $d(\star J) = 0$  follows purely from the algebraic identity  $d^2 = 0$  applied to Axiom 2.

**The Noether interpretation.** The algebraic derivation above has a physical reading through Noether's theorem, which provides deeper context. The standard electromagnetic action is:

$$S = \int \left( \frac{1}{2} F \wedge \star F + A \wedge \star J \right)$$

The gauge transformation  $A \rightarrow A + d\chi$  is a  $U(1)$  symmetry of this action (Section 4), and Noether's theorem associates a conserved current to every continuous symmetry. The conserved current associated with gauge invariance is precisely  $J^\mu$ . Charge conservation is therefore the **Noether invariant** of the  $U(1)$  gauge symmetry.

The two perspectives are complementary:

- **Algebraic:**  $d(\star J) = 0$  follows from  $d^2 = 0$  in two steps. No action needed.
- **Variational:**  $\partial_\mu J^\mu = 0$  is the Noether current of the  $U(1)$  gauge symmetry of the action  $S$ .

The algebraic derivation is logically prior (it requires fewer assumptions), but the Noether interpretation explains **why** this conservation law exists: it is the shadow of gauge symmetry.

In kernel language, charge conservation is a consequence of the **nilpotency** of  $K_{\text{em}}$ . Applying  $d$  to the image of  $K_{\text{em}}$  — i.e., composing  $K_{\text{em}}$  with itself — yields zero:  $d(d(\star F)) = 0$ . The constraint  $d(\star J) = 0$  is therefore forced by the kernel’s self-composition property. This is a structural feature of the electromagnetic sector that the POT framework makes explicit: the projection kernel’s nilpotency constrains the source.

Charge is not an arbitrary ‘stuff’ assigned to particles. It is a geometric quantity forced into existence by the structure of the exterior algebra. The source term  $J$  in  $d\star F = \star J$  is not a free choice — it is constrained by  $d(\star J) = 0$ .

This is the closing argument of the paper: two axioms, the background structure of the Lorentzian manifold, and the nilpotency of  $d$  yield the complete classical electrodynamics. The field equations, the gauge symmetry, the topological structure, and the conservation laws are all derivable. The only irreducible physical content is concentrated in a single equation:  $d\star F = \star J$ .

## 8 Einstein–Maxwell Coupling

The electromagnetic field is not isolated; it couples to gravity through the stress-energy tensor. The Einstein–Maxwell field equation is:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}^{\text{EM}}$$

where  $G_{\mu\nu}$  is the Einstein tensor,  $\Lambda$  the cosmological constant,  $\kappa = 8\pi\frac{G}{c^4}$  Einstein’s gravitational constant, and  $T_{\mu\nu}^{\text{EM}}$  the electromagnetic stress-energy tensor.

In the Kleis standard library, this equation is axiomatized in the `EinsteinMaxwell` structure, which also enforces the symmetry properties:  $F_{\mu\nu} = -F_{\nu\mu}$  (antisymmetry of the field tensor),  $T_{\mu\nu}^{\text{EM}} = T_{\nu\mu}^{\text{EM}}$  (symmetry of the stress-energy tensor), and  $g_{\mu\nu} = g_{\nu\mu}$  (symmetry of the metric).

The coupling has a structural parallel in the Cartan formalism. Both gravity and electromagnetism are built from the **same** mathematical operations:

- **Gravity:** the curvature 2-form  $R_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c$  is a matrix of 2-forms built from the connection  $\omega$  via the exterior derivative  $d$  and wedge product  $\wedge$ .
- **Electromagnetism:** the field strength 2-form  $F = dA$  is a single 2-form built from the potential  $A$  via the same exterior derivative  $d$ .

The connection  $\omega_b^a$  and the potential  $A$  are both 1-forms. The curvature  $R_b^a$  and the field strength  $F$  are both 2-forms. The Bianchi identity  $dR + \omega \wedge R - R \wedge \omega = 0$  and the homogeneous Maxwell equation  $dF = 0$  are both consequences of  $d^2 = 0$ .

We emphasize that this is a **structural parallel**, not a derivation of electrodynamics from gravity or vice versa. The Einstein–Maxwell equation couples the two sectors, but the coupling is an additional axiom (relating  $G_{\mu\nu}$  to  $T_{\mu\nu}^{\text{EM}}$ ), not a consequence of the exterior calculus alone. What **is** a consequence of the shared algebraic structure is that the conservation law  $\nabla_\mu T_{\text{EM}}^{\mu\nu} = 0$  follows from the Bianchi identity  $\nabla_\mu G^{\mu\nu} = 0$  once the coupling is assumed.

In the POT kernel framework, both sectors use admissible kernels built from the same Cartan substrate. The electromagnetic kernel  $K_{\text{em}} = d$  is admissible (linear). The gravitational sector uses the Cartan machinery ( $\omega \rightarrow d\omega + \omega \wedge \omega$ ), which is **not** admissible due to the quadratic  $\omega \wedge \omega$  term — but the linearized gravitational perturbation theory **is** admissible, and the composition closure axiom (`compose_admissible`) guarantees that combinations of admissible sectors remain admissible. The operations are shared; the physical content is sector-specific; and the kernel framework provides the unifying vocabulary.

## 9 The Yang–Mills Generalization

Electrodynamics is the  $U(1)$  gauge theory. The natural generalization to non-abelian gauge groups — Yang–Mills theory — replaces the abelian field strength  $F = dA$  with:

$$F = dA + A \wedge A$$

The additional  $A \wedge A$  term (absent in electrodynamics because the  $U(1)$  Lie algebra is abelian) encodes the self-interaction of the gauge field. The Bianchi identity becomes the gauge-covariant version:

$$D_A F = dF + [A, F] = 0$$

where  $D_A$  is the gauge-covariant exterior derivative.

This structure is **already formalized** in the Kleis standard library as the `YangMillsForm` structure, with the axioms:

- $F = dA + A \wedge A$  (curvature definition)
- $D_A F = 0$  (Bianchi identity)

The formal pattern is identical to the Cartan geometry of gravity:

- **Gravity:** Connection  $\omega$ , curvature  $R = d\omega + \omega \wedge \omega$ , Bianchi identity  $D_\omega R = 0$ .
- **Yang–Mills:** Connection  $A$ , curvature  $F = dA + A \wedge A$ , Bianchi identity  $D_A F = 0$ .

The gauge group determines the physics:  $U(1)$  gives electrodynamics,  $SU(2) \times U(1)$  gives the electroweak theory, and  $SU(3)$  gives quantum chromodynamics. But the mathematical structure — connection, curvature, Bianchi identity — is the same in every case.

In the POT kernel framework, this generalization has a precise consequence: **the Yang–Mills kernel is not admissible**. The map  $A \rightarrow dA + A \wedge A$  violates `kernel_lin_add` because the  $A \wedge A$  term is quadratic:  $d(A + B) + (A + B) \wedge (A + B) \neq (dA + A \wedge A) + (dB + B \wedge B)$  due to cross terms  $A \wedge B + B \wedge A$ . Electrodynamics is therefore distinguished within gauge theory as the **unique sector with an admissible projection kernel** — the abelian boundary case where the self-interaction vanishes and linearity survives.

This classification is a genuinely POT-specific result. Standard gauge theory classifies theories by gauge group ( $U(1)$ ,  $SU(2)$ ,  $SU(3)$ , etc.). The POT framework adds a structural classification: by whether the projection kernel satisfies the admissibility axioms. Only  $U(1)$  does. Non-abelian gauge theories require extending the kernel framework beyond strict linearity — a direction for future work that may connect to the existing Yang–Mills axioms in the standard library.

## 10 Discussion and Conclusion

We have shown that, given the background structure of a smooth oriented Lorentzian 4-manifold with its exterior algebra and Hodge star, the complete differential-form structure of classical electrodynamics follows from two axioms and the nilpotency  $d^2 = 0$ . The derivation chain is short:

Theorem	Derivation	Steps from axioms
Gauge invariance	$d(A + d\chi) = dA + d^2\chi = dA$	1 step
Homogeneous Maxwell ( $dF = 0$ )	$dF = d(dA) = d^2A = 0$	1 step
Charge conservation ( $d(\star J) = 0$ )	$d(\star J) = d(d\star F) = d^2(\star F) = 0$	2 steps
No monopoles	Topological: $H_{\text{dR}}^2(\mathbb{R}^4) = 0$	Cohomology
Poynting / energy conservation	Bianchi identity + Einstein–Maxwell	Structural

The irreducible physical content is concentrated in a single dynamical equation:  $d\star F = \star J$ . This is the **only** statement that connects the geometry (the field  $F$ ) to the physical world (the source current  $J$ ). Everything else is either mathematical infrastructure (manifold, metric, exterior algebra) or a derivable consequence thereof.

Several features of this result merit comment.

**The role of  $d^2 = 0$ .** The nilpotency of the exterior derivative is not a physical assumption. It is a structural property of the exterior algebra on any smooth manifold, equivalent to the symmetry of mixed partial derivatives (Schwarz’s theorem). Every derivation in this paper reduces, ultimately, to this single identity. Gauge invariance, the homogeneous Maxwell equation, and charge conservation are all shadows of  $d^2 = 0$  projected through different algebraic paths.

**Machine verification as methodology.** The Kleis platform makes it possible to treat physics derivations as executable programs: the axioms are inputs, the derivations are computations, and the theorems are outputs checked by an independent solver. This is not merely a presentation choice. It enforces a discipline that informal derivations do not: every step must be justified by an explicit axiom, every axiom must be stated, and the dependency graph — which theorem follows from which axiom in how many steps — is a machine-checkable artifact. The derivation table above is the output of this process, not a human summary of it. We believe this represents a productive mode of interaction between formal methods and theoretical physics: not replacing physical intuition, but making its logical structure auditable.

**What is and is not claimed.** The formulation of electrodynamics in terms of differential forms is well known; the key references are Flanders, Misner–Thorne–Wheeler, Frankel, and Nakahara. Our contribution is not the mathematics, which is standard, but: (a) the isolation of a **minimal** axiom set and a complete **dependency graph** showing which results follow from which axioms in how many steps; (b) the **machine verification** of every derivation, making the axiom dependencies checkable rather than informal; (c) the identification of the exterior derivative as an admissible kernel in the POT sense, with gauge equivalence as projective equivalence and gauge

orbits as the kernel’s nullspace; and (d) the classification of electrodynamics as the unique gauge sector with an admissible projection kernel (the abelian boundary case).

**Comparison with the standard presentation.** Textbooks typically present Maxwell’s equations as four independent laws (Gauss, Gauss for magnetism, Faraday, Ampère–Maxwell), then derive gauge invariance and charge conservation as consequences. Our presentation inverts this: we start with the definition  $F = dA$  and the single inhomogeneous equation  $d \star F = \star J$ , and derive everything else. The ‘four laws’ are a redundant encoding of two independent statements.

**The Lorentzian Hodge star.** The sign in the involutivity relation  $\star \star \alpha = (-1)^{p(n-p)+s} \alpha$  depends on the metric signature  $s$  (number of negative eigenvalues). For the electromagnetic 2-form on Minkowski spacetime,  $\star \star F = -F$ . Getting this sign wrong would reverse the sign of the energy-momentum tensor, predicting negative energy densities. The parameterization of the Hodge star by the signature integer  $s$  is therefore not a mathematical nicety but a physical necessity.

**Connection to the POT series.** This paper is the third in a series demonstrating that the admissible kernel framework of Projected Ontology Theory unifies diverse physical sectors under a common formal structure. The first paper identified the gravitational kernel (logarithmic Green’s function, slow-decay coherence) and derived flat rotation curves. The second identified the measurement kernel (spinor projections parameterized by detector angle) and derived Bell violation. The present paper identifies the electromagnetic kernel ( $d|_{\Omega^1}$ ) and derives Maxwell’s equations. In each case, the kernel is admissible, the projection is lossy (many pre-observable states map to the same observable), and the nullspace characterizes the redundancy (gravitational gauge freedom, measurement basis choice, electromagnetic gauge orbits). The composition closure axiom (`compose_admissible`) guarantees that the combined multi-sector kernel is admissible, providing a formal basis for the Einstein–Maxwell coupling and future unification with the measurement sector.

**Toward verified unified field theory.** The Yang–Mills generalization  $F = dA + A \wedge A$  is already axiomatized. The formal structure is identical to gravity (connection, curvature, Bianchi identity). The gauge group determines the physics; the mathematics is universal. A verified treatment of the Standard Model gauge sector —  $SU(3) \times SU(2) \times U(1)$  — is within reach using the same derivation patterns and the same verification infrastructure.

We conclude with a philosophical observation. In the Projected Ontology framework, the distinction between ‘law’ and ‘theorem’ depends on whether a statement carries irreducible physical content or follows from the structure of the admissible kernel. For electrodynamics, the inventory is clear: one dynamical equation ( $d \star F = \star J$ ), one definitional axiom ( $F = dA$ ), and a background manifold. Everything else — gauge invariance (projective equivalence), the homogeneous Maxwell equation (kernel nilpotency), charge conservation (nilpotency applied to the source constraint), the absence of monopoles on trivial bundles (surjectivity of the kernel) — is theorem. The universe does not ‘obey’ four independent Maxwell equations; it satisfies one dynamical constraint on a Lorentzian manifold, and the rest is the compiled output of an admissible projection kernel.

## 11 Appendix A: Standard Library Axiom Inventory

The following table lists every axiom from the Kleis standard library used in the derivations of this paper.

Axiom	Structure / File	Statement
$d^2 = 0$	ExteriorDerivative / differential_forms.kleis	$d(d\alpha) = 0$ for any $p$ -form $\alpha$
Leibniz rule	ExteriorDerivative	$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta)$
Hodge involutivity	HodgeStar / differential_forms.kleis	$\star \star \alpha = (-1)^{p(n-p)+s} \alpha$
Lorentzian Hodge	LorentzianHodge	$\star \star \alpha = (-1)^{p(n-p)+1} \alpha$ (specialization with $s = 1$ )
Interior nilpotency	InteriorProduct	$\iota_X \iota_X = 0$
Cartan Magic Formula	CartanCalculus	$\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d$
$F = dA$	ElectromagneticForm	Field strength from potential (Axiom 1)
$d \star F = \star J$	ElectromagneticForm	Inhomogeneous Maxwell (Axiom 2)
Exact $\Rightarrow$ Closed	DeRhamCohomology	$d\beta = 0 \Rightarrow d(d\beta) = 0$ (exact implies closed)
$F$ antisymmetry	FieldTensorProperties / maxwell.kleis	$F_{\mu\nu} = -F_{\nu\mu}$
$T^{\text{EM}}$ symmetry	EinsteinMaxwell / maxwell.kleis	$T_{\mu\nu}^{\text{EM}} = T_{\nu\mu}^{\text{EM}}$
Einstein–Maxwell	EinsteinMaxwell	$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}^{\text{EM}}$
Bianchi identity	CurvatureForm / cartan_geometry.kleis	$dR_b^a + \omega_c^a \wedge R_b^c - R_c^a \wedge \omega_b^c = 0$
Yang–Mills curvature	YangMillsForm / differential_forms.kleis	$F = dA + A \wedge A$
Yang–Mills Bianchi	YangMillsForm	$D_A F = 0$
Kernel linearity (add)	AdmissibleKernel / pot_admissible_kernels.kleis	$K(G, a+b) = K(G, a) + K(G, b)$ when admissible
Kernel linearity (smul)	AdmissibleKernel	$K(G, c \cdot a) = c \cdot K(G, a)$ when admissible
Kernel zero	AdmissibleKernel	$K(G, 0) = 0$ when admissible
Projective equiv.	ProjectiveEquivalence	$a \sim b$ iff $K(G, a) = K(G, b)$
Nullspace	Nullspace	$a \in \ker(K)$ iff $K(G, a) = 0$ ; closed under add/smul

Composition closure	KernelComposition	admissible $G_1, G_2 \Rightarrow$ admissible $G_1 \circ G_2$
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